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MATHEMATICS FOR THE  
NONMATHEMATICIAN

# MATHEMATICS FOR THE NONMATHEMATICIAN

MORRIS KLINE

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image

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PREFACE

“ . . . I consider that without understanding as much of the abstruser part of geometry, as Archimedes or Apollonius, one may understand enough to be assisted by it in the contemplation of nature; and that one needs not know the profoundest mysteries of it to be able to discern its usefulness. . . . I have often wished that I had employed about the speculative part of geometry, and the cultivation of the specious [symbolic] algebra I had been taught very young, a good part of that time and industry that I spent about surveying and fortification. . . .”

ROBERT BOYLE

I believe as firmly as I have in the past that a mathematics course addressed to liberal arts students must present the scientific and humanistic import of the subject. Whereas mathematics proper makes little appeal and seems even less pointed to most of these students, the subject becomes highly significant to them when it is presented in a cultural context. In fact, the branches of elementary mathematics were created primarily to serve extra-mathematical needs and interests. In the very act of meeting such needs each of these creations has proved to have inestimable importance for man’s understanding of the nature of his world and himself.

That so many professors have chosen to teach mathematics as an integral part of Western culture, as evidenced by their reception of my earlier book, Mathematics: A Cultural Approach, has been extremely gratifying. That book will continue to be available. In the present revision and abridgment, which has been designed to meet the needs of particular groups of students, the spirit of the original text has been preserved. The historical approach has been retained because it is intrinsically interesting, provides motivation for the introduction of various topics, and gives coherence to the body of material. Each topic or branch of mathematics dealt with is shown to be a response to human interests, and the cultural import of the technical development is presented. I adhered to the principle that the level of rigor should be suited to the mathematical age of the student rather than to the age of mathematics.

As in the earlier text, several of the topics are treated quite differently from what is now fashionable. These are the real number system, logic, and set theory. I tried to present these topics in a context and with a level of emphasis which I believe to be appropriate for an elementary course in mathematics. In this book, the axiomatic approach to the real numbers is formulated after the various types of numbers and their properties are derived from physical situations and uses. The treatment of logic is confined to the fundamentals of Aristotelian logic. And set theory serves as an illustration of a different kind of algebra.

The changes made in this revision are intended to suit special groups. Some students need more review and drill on elementary concepts and techniques than the earlier book provides. Others, chiefly those preparing for teaching on the elementary level, need to learn more about elementary mathematics than their high school courses covered. Teachers of twelfth-year high school courses and one-semester college courses often found the extensive amount of material in Mathematics: A Cultural Approach rather disconcerting because it offered so much more than could be covered.

To meet the needs of these groups I have made the following changes:

1. Four of the chapters devoted entirely to cultural influences have been dropped. The size of the original book has thereby been reduced considerably.

2. A few applications of mathematics to science have been omitted, primarily to reduce the size of the text.

3. Some of the chapters on technical topics, [Chapter 3](#ch3) on logic and mathematics, [Chapter 4](#ch4) on number, [Chapter 5](#ch5) on elementary algebra, and [Chapter 21](#ch21) on arithmetics and their algebras have been expanded.

4. Additional drill exercises have been added within a few chapters, and a set of review exercises providing practice in technique has been added to each of a number of chapters.

5. Improvements in presentation have been made in a number of places.

With respect to use in courses, it is probably true of the present text, as it is of the earlier one, that it contains more material than can be covered in some courses. However, many of the chapters as well as sections in chapters are not essential to the logical continuity. These chapters and sections have been starred image. Thus [Chapter 10](#ch10) on painting shows historically how mathematicians were led to projective geometry ([Chapter 11](#ch11)), but from a logical standpoint, [Chapter 10](#ch10) is not needed in order to understand the succeeding chapter. [Chapter 19](#ch19) on musical sounds is an application of the material on the trigonometric functions in [Chapter 18](#ch18) but is not essential to the continuity. The two chapters on the calculus are not used in the succeeding chapters. Desirable as it may be to give students some idea of what the calculus is about, it may still be necessary in some classes to omit these chapters. The same can be said of the chapters on statistics ([Chapter 22](#ch22)) and probability ([Chapter 23](#ch23)).

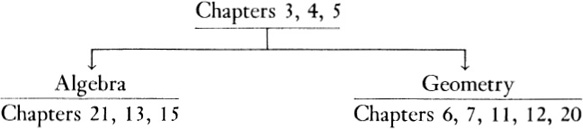
As for sections within chapters, [Chapter 6](#ch6) on Euclidean geometry may well serve as an illustration. The mathematical material of this chapter is intended as a review of some basic ideas and theorems of Euclidean geometry and as an introduction to the conic sections. Some of the familiar applications are given in [Section 6-3](#ch6_3) (see the Table of Contents) and probably should be taken up. However the applications to light in [Sections 6-4](#ch6_4) and [6-6](#ch6_6) and the discussion of cultural influences in [Section 6-7](#ch6_7) can be omitted.

Some of the material, whether or not included in the following recommendations for particular groups, can be left to student reading. In fact, the first two chapters were deliberately fashioned so that they could be read by students. The objective here, in addition to presenting intrinsically important ideas, was to induce students to read a mathematics book, to give them the confidence to do so, and to get them into the habit of doing so. It seems necessary to counter the students’ impression, resulting no doubt from elementary and high school instruction in mathematics, that whereas history texts are to be read, mathematics texts are essentially reference books for formulas and homework exercises.

For courses emphasizing the number concept and its extension to algebra, it is possible to take advantage of the logical independence of numerous chapters and use [Chapters 3](#ch3) through [5](#ch5) on reasoning, arithmetic, and algebra and [Chapter 21](#ch21) on different algebras. To pursue the development of this theme into the area of functions one can include [Chapters 13](#ch13) and [15](#ch15).

Courses emphasizing geometry can utilize [Chapters 6](#ch6), [7](#ch7), [11](#ch11), [12](#ch12), and [20](#ch20) on Euclidean geometry, trigonometry, projective geometry, coordinate geometry, and non-Euclidean geometry respectively. Some algebra, that reviewed in [Chapter 5](#ch5), is involved in [Chapters 7](#ch7) and [12](#ch12). If knowledge of the material of [Chapter 5](#ch5) cannot be presupposed, this chapter must precede the treatment of geometry.

The essence of the two preceding suggestions may be diagrammed thus:



Of course, starred sections in these chapters are optional.

For a one-semester liberal arts course, the basic content can be as follows:

|  |  |
| --- | --- |
| [Chapter 2](#ch2) | on a historical orientation, |
| [Chapter 3](#ch3) | on logic and mathematics, |
| [Chapters 4](#ch4) and [5](#ch5) | on the number system and elementary algebra, |
| [Chapter 6](#ch6) | through [Section 6–5](#ch6_5), on Euclidean geometry, |
| [Chapter 7](#ch7) | through [Section 7–3](#ch7_3), on trigonometry, |
| [Chapter 12](#ch12) | on coordinate geometry, |
| [Chapter 13](#ch13) | on functions and their uses, |
| [Chapter 14](#ch14) | through [Section 14–4](#ch14_4), on parametric equations, |
| [Chapter 15](#ch15) | through [Section 15–10](#ch15_10), on the further use of functions in science, |
| [Chapter 20](#ch20) | on non-Euclidean geometry, |
| [Chapter 21](#ch21) | on different algebras. |

Any additional material would enrich the course but would not be needed for continuity.

Though the teacher’s problem of presenting material outside the domain of mathematics proper is far simpler with this text than with the earlier one, it may still be necessary to assure him that he need not hesitate to undertake this task. The feeling that one must be an authority in a subject to say anything about it is unfounded. We are all laymen outside the field of our own specialty, and we should not be ashamed to point this out to students. In contiguous areas we are merely giving indications of ideas that students may pursue further in other courses or in independent reading.

I hope that this text will serve the needs of the groups of students to which it is addressed and that, despite the somewhat greater emphasis on technical matters, it will convey the rich significance of mathematics.

I wish to thank my wife Helen for her critical scrutiny of the contents and her conscientious reading of the proofs. I wish to express, also, my thanks to members of the Addison-Wesley staff for very helpful suggestions and for their continuing support of a culturally oriented approach to mathematics.

|  |  |
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| New York, 1967 | M.K. |

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MATHEMATICS FOR THE  
NONMATHEMATICIAN

## CHAPTER 1

WHY MATHEMATICS?

In mathematics I can report no deficience, except it be that men do not sufficiently understand the excellent use of the Pure Mathematics. . . .

FRANCIS BACON

One can wisely doubt whether the study of mathematics is worth while and can find good authority to support him. As far back as about the year 400 A.D., St. Augustine, Bishop of Hippo in Africa and one of the great fathers of Christianity, had this to say:

The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell.

Perhaps St. Augustine, with prophetic insight into the conflicts which were to arise later between the mathematically minded scientists of recent centuries and religious leaders, was seeking to discourage the further development of the subject. At any rate there is no question as to his attitude.

At about the same time that St. Augustine lived, the Roman jurists ruled, under the Code of Mathematicians and Evil-Doers, that “to learn the art of geometry and to take part in public exercises, an art as damnable as mathematics, are forbidden.”

Even the distinguished seventeenth-century contributor to mathematics, Blaise Pascal, decided after studying mankind that the pure sciences were not suited to it. In a letter to Fermat written on August 10, 1660, Pascal says: “To speak freely of mathematics, I find it the highest exercise of the spirit; but at the same time I know that it is so useless that I make little distinction between a man who is only a mathematician and a common artisan. Also, I call it the most beautiful profession in the world; but it is only a profession; and I have often said that it is good to make the attempt [to study mathematics], but not to use our forces: so that I would not take two steps for mathematics, and I am confident that you are strongly of my opinion.” Pascal’s famous injunction was, “Humble thyself, impotent reason.”

The philosopher Arthur Schopenhauer, who despised mathematics, said many nasty things about the subject, among others that the lowest activity of the spirit is arithmetic, as is shown by the fact that it can be performed by a machine. Many other great men, for example, the poet Johann Wolfgang Goethe and the historian Edward Gibbon, have felt likewise and have not hesitated to express themselves. And so the student who dislikes the subject can claim to be in good, if not living, company.

In view of the support he can muster from authorities, the student may well inquire why he is asked to learn mathematics. Is it because Plato, some 2300 years ago, advocated mathematics to train the mind for philosophy? Is it because the Church in medieval times taught mathematics as a preparation for theological reasoning? Or is it because the commercial, industrial, and scientific life of the Western world needs mathematics so much? Perhaps the subject got into the curriculum by mistake, and no one has taken the trouble to throw it out. Certainly the student is justified in asking his teacher the very question which Mephistopheles put to Faust:

Is it right, I ask, is it even prudence,

To bore thyself and bore the students?

Perhaps we should begin our answers to these questions by pointing out that the men we cited as disliking or disapproving of mathematics were really exceptional. In the great periods of culture which preceded the present one, almost all educated people valued mathematics. The Greeks, who created the modern concept of mathematics, spoke unequivocally for its importance. During the Middle Ages and in the Renaissance, mathematics was never challenged as one of the most important studies. The seventeenth century was aglow not only with mathematical activity but with popular interest in the subject. We have the instance of Samuel Pepys, so much attracted by the rapidly expanding influence of mathematics that at the age of thirty he could no longer tolerate his own ignorance and begged to learn the subject. He began, incidentally, with the multiplication table, which he subsequently taught to his wife. In 1681 Pepys was elected president of the Royal Society, a post later held by Isaac Newton.

In perusing eighteenth-century literature, one is struck by the fact that the journals which were on the level of our Harper’s and the Atlantic Monthly contained mathematical articles side by side with literary articles. The educated man and woman of the eighteenth century knew the mathematics of their day, felt obliged to be au courant with all important scientific developments, and read articles on them much as modern man reads articles on politics. These people were as much at home with Newton’s mathematics and physics as with Pope’s poetry.

The vastly increased importance of mathematics in our time makes it all the more imperative that the modern person know something of the nature and role of mathematics. It is true that the role of mathematics in our civilization is not always obvious, and the deeper and more complex modern applications are not readily comprehended even by specialists. But the essential nature and accomplishments of the subject can still be understood.

Perhaps we can see more easily why one should study mathematics if we take a moment to consider what mathematics is. Unfortunately the answer cannot be given in a single sentence or a single chapter. The subject has many facets or, some might say, is Hydra-headed. One can look at mathematics as a language, as a particular kind of logical structure, as a body of knowledge about number and space, as a series of methods for deriving conclusions, as the essence of our knowledge of the physical world, or merely as an amusing intellectual activity. Each of these features would in itself be difficult to describe accurately in a brief space.

Because it is impossible to give a concise and readily understandable definition of mathematics, some writers have suggested, rather evasively, that mathematics is what mathematicians do. But mathematicians are human beings, and most of the things they do are uninteresting and some, embarrassing to relate. The only merit in this proposed definition of mathematics is that it points up the fact that mathematics is a human creation.

A variation on the above definition which promises more help in understanding the nature, content, and values of mathematics, is that mathematics is what mathematics does. If we examine mathematics from the standpoint of what it is intended to and does accomplish, we shall undoubtedly gain a truer and clearer picture of the subject.

Mathematics is concerned primarily with what can be accomplished by reasoning. And here we face the first hurdle. Why should one reason? It is not a natural activity for the human animal. It is clear that one does not need reasoning to learn how to eat or to discover what foods maintain life. Man knew how to feed, clothe, and house himself millenniums before mathematics existed. Getting along with the opposite sex is an art rather than a science mastered by reasoning. One can engage in a multitude of occupations and even climb high in the business and industrial world without much use of reasoning and certainly without mathematics. One’s social position is hardly elevated by a display of his knowledge of trigonometry. In fact, civilizations in which reasoning and mathematics played no role have endured and even flourished. If one were willing to reason, he could readily supply evidence to prove that reasoning is a dispensable activity.

Those who are opposed to reasoning will readily point out other methods of obtaining knowledge. Most people are in fact convinced that their senses are really more than adequate. The very common assertion “seeing is believing” expresses the common reliance upon the senses. But everyone should recognize that the senses are limited and often fallible and, even where accurate, must be interpreted. Let us consider, as an example, the sense of sight. How big is the sun? Our eyes tell us that it is about as large as a rubber ball. This then is what we should believe. On the other hand, we do not see the air around us, nor for that matter can we feel, touch, smell, or taste it. Hence we should not believe in the existence of air.

To consider a somewhat more complicated situation, suppose a teacher should hold up a fountain pen and ask, What is it? A student coming from some primitive society might call it a shiny stick, and indeed this is what the eyes see. Those who call it a fountain pen are really calling upon education and experience stored in their minds. Likewise, when we look at a tall building from a distance, it is experience which tells us that the building is tall. Hence the old saying that “we are prone to see what lies behind our eyes, rather than what appears before them.”

Every day we see the sun where it is not. For about five minutes before what we call sunset, the sun is actually below the geometrical horizon and should therefore be invisible. But the rays of light from the sun curve toward us as they travel in the earth’s atmosphere, and the observer at P ([Fig. 1–1](#fig1_1)) not only “sees” the sun but thinks the light is coming from the direction O′P. Hence he believes the sun is in that direction.

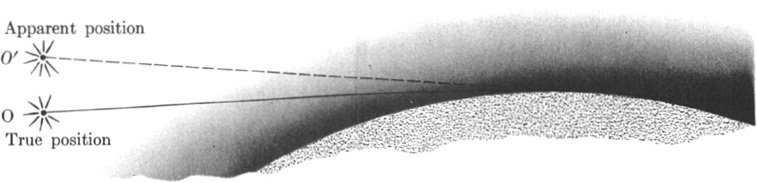


Fig. 1–1.  
Deviation of a ray by the earth's atmosphere.

The senses are obviously helpless in obtaining some kinds of knowledge, such as the distance to the sun, the size of the earth, the speed of a bullet (unless one wishes to feel its velocity), the temperature of the sun, the prediction of eclipses, and dozens of other facts.

If the senses are inadequate, what about experimentation or, in simple cases, measurement? One can and in fact does learn a great deal by such means. But suppose one wants to find a very simple quantity, the area of a rectangle. To obtain it by measurement, one could lay off unit squares to cover the area and then count the number of squares. It is at least a little simpler to measure the lengths of the sides and then use a formula obtained by reasoning, namely, that the area is the product of length and width. In the only slightly more complicated problem of determining how high a projectile will go, we should certainly not consider traveling with the projectile.

As to experimentation, let us consider a relatively simple problem of modern technology. One wishes to build a bridge across a river. How long and how thick should the many beams be? What shape should the bridge take? If it is to be supported by cables, how long and how thick should these be? Of course one could arbitrarily choose a number of lengths and thicknesses for the beams and cables and build the bridge. In this event, it would only be fair that the experimenter be the first to cross this bridge.

It may be clear from this brief discussion that the senses, measurement, and experimentation, to consider three alternative ways of acquiring knowledge, are by no means adequate in a variety of situations. Reasoning is essential. The lawyer, the doctor, the scientist, and the engineer employ reasoning daily to derive knowledge that would otherwise not be obtainable or perhaps obtainable only at great expense and effort. Mathematics more than any other human endeavor relies upon reasoning to produce knowledge.

One may be willing to accept the fact that mathematical reasoning is an effective procedure. But just what does mathematics seek to accomplish with its reasoning? The primary objective of all mathematical work is to help man study nature, and in this endeavor mathematics cooperates with science. It may seem, then, that mathematics is merely a useful tool and that the real pursuit is science. We shall not attempt at this stage to separate the roles of mathematics and science and to evaluate the relative merits of their contributions. We shall simply state that their methods are different and that mathematics is at least an equal partner with science.

We shall see later how observations of nature are framed in statements called axioms. Mathematics then discloses by reasoning secrets which nature may never have intended to reveal. The determination of the pattern of motion of celestial bodies, the discovery and control of radio waves, the understanding of molecular, atomic, and nuclear structures, and the creation of artificial satellites are a few basically mathematical achievements. Mathematical formulation of physical data and mathematical methods of deriving new conclusions are today the substratum in all investigations of nature.

The fact that mathematics is of central importance in the study of nature reveals almost immediately several values of this subject. The first is the practical value. The construction of bridges and skyscrapers, the harnessing of the power of water, coal, electricity, and the atom, the effective employment of light, sound, and radio in illumination, communication, navigation, and even entertainment, and the advantageous employment of chemical knowledge in the design of materials, in the production of useful forms of oil, and in medicine are but a few of the practical achievements already attained. And the future promises to dwarf the past.

However, material progress is not the most compelling reason for the study of nature, nor have practical results usually come about from investigations so directed. In fact, to overemphasize practical values is to lose sight of the greater significance of human thought. The deeper reason for the study of nature is to try to understand the ways of nature, that is, to satisfy sheer intellectual curiosity. Indeed, to ask disinterested questions about nature is one of the distinguishing marks of mankind. In all civilizations some people at least have tried to answer such questions as: How did the universe come about? How old is the universe and the earth in particular? How large are the sun and the earth? Is man an accident or part of a larger design? Will the solar system continue to function or will the earth some day fall into the sun? What is light? Of course, not all people are interested in such questions. Food, shelter, sex, and television are enough to keep many happy. But others, aware of the pervasive natural mysteries, are more strongly obsessed to resolve them than any business man is to acquire wealth and power.

Beyond improvement in the material life of man and beyond satisfaction of intellectual curiosity, the study of nature offers intangible values of another sort, especially the abolition of fear and terror and their replacement by a deep, quiet satisfaction in the ways of nature. To the uneducated and to those uninitiated in the world of science, many manifestations of nature have appeared to be agents of destruction sent by angry gods. Some of the beliefs in ancient and even medieval Europe may be of special interest in view of what happened later. The sun was the center of all life. As winter neared and the days became shorter, the people believed that a battle between the gods of light and darkness was taking place. Thus the god Wodan was supposed to be riding through heaven on a white horse followed by demons, all of whom sought every opportunity to harm people. When, however, the days began to lengthen and the sun began to show itself higher in the sky each day, the people believed that the gods of light had won. They ceased all work and celebrated this victory. Sacrifices were offered to the benign gods. Symbols of fertility such as fruit and nuts, whose growth is, of course, aided by the sun, were placed on the altars. To symbolize further the desire for light and the joy in light, a huge log was placed in the fire to burn for twelve days, and candles were lit to heighten the brightness.

The beliefs and superstitions which have been attached to events we take in stride are incredible to modern man. An eclipse of the sun, a threat to the continuance of the light and heat which causes crops to grow, meant that the heavenly body was being swallowed up by a dragon. Many Hindu people believe today that a demon residing in the sky attacks the sun once in a while and that this is what causes the eclipse. Of course, when prayers, sacrifices, and ceremonies were followed by the victory of the sun or moon, it was clear that these rituals were the effective agent and so had to be pursued on every such occasion. In addition, special magic potions drunk during eclipses insured health, happiness, and wisdom.

To primitive peoples of the past, thunder, lightning, and storms were punishments visited by the gods on people who had apparently sinned in some way. The stories in the Old Testament of the flood and of the destruction of Sodom and Gomorrah by fire and brimstone are examples of such acts of wrath by the God of the Hebrews. Hence there was continual concern and even dread about what the gods might have in mind for helpless humans. The only recourse was to propitiate the divine powers, so that they would bring good fortune instead of evil.

Fears, dread, and superstitions have been eliminated, at least in our Western civilization, by just those intellectually curious people who have studied nature’s mighty displays. Those “seemingly unprofitable amusements of speculative brains” have freed us from serfdom, given us undreamed of powers, and, in fact, have replaced negative doctrines by positive mathematical laws which reveal a remarkable order and uniformity in nature. Man has emerged as the proud possessor of knowledge which has enabled him to view nature calmly and objectively. An eclipse of the sun occurring on schedule is no longer an occasion for trembling but for quiet satisfaction that we know nature’s ways. We breathe freely, knowing that nature will not be willful or capricious.

Indeed, man has been remarkably successful in his study of nature. History is said to repeat itself, but, in general, the circumstances of the supposed repetition are not the same as those of the earlier occurrence. As a consequence, the history of man has not been too effective a guide for the future. Nature is kinder. When nature repeats herself, and she does so constantly, the repetitions are exact facsimiles of previous events, and therefore man can anticipate nature’s behavior and be prepared for what will take place. We have learned to recognize the patterns of nature and we can speak today of the uniformity of nature and delight in the regularity of her behavior.

The successes of mathematics in the study of inanimate nature have inspired in recent times the mathematical study of human nature. Mathematics has not only contributed to the very practical institutions such as banking, insurance, pension systems, and the like, but it has also supplied some substance, spirit, and methodology to the infant sciences of economics, politics, and sociology. Number, quantitative studies, and precise reasoning have replaced vague, subjective, and ineffectual speculations and have already given evidence of greater values to come.

As man turns to thoughts about himself and his fellow man, other questions occur to him which are as fundamental as any he can ask. Why is man born? What purposes does he serve or should he serve? What future awaits him? The knowledge acquired about our physical universe has profound implications for the origin and role of man. Moreover, as mathematics and science have amassed increasing knowledge and power, they have gradually encompassed the biological and psychological sciences, which in turn have shed further light on man’s physical and mental life. Thus it has come about that mathematics and science have profoundly affected philosophy and religion.

Perhaps the most profound questions in the realm of philosophy are, What is truth and how does man acquire it? Though we have no final answer to these questions, the contribution of mathematics toward this end is paramount. For two millenniums mathematics was the prime example of truths man had unearthed. Hence all investigations of the problem of acquiring truths necessarily reckoned with mathematics. Though some startling developments in the nineteenth century altered completely our understanding of the nature of mathematics, the effectiveness of the subject, especially in representing and analyzing natural phenomena, has still kept mathematics the focal point of all investigations into the nature of knowledge. Not the least significant aspect of this value of mathematics has been the insight it has given us into the ways and powers of the human mind. Mathematics is the supreme and most remarkable example of the mind’s power to cope with problems, and as such it is worthy of study.

Among the values which mathematics offers are its services to the arts. Most people are inclined to believe that the arts are independent of mathematics, but we shall see that mathematics has fashioned major styles of painting and architecture, and the service mathematics renders to music has not only enabled man to understand it, but has spread its enjoyment to all corners of our globe.

Practical, scientific, philosophical, and artistic problems have caused men to investigate mathematics. But there is one other motive which is as strong as any of these — the search for beauty. Mathematics is an art, and as such affords the pleasures which all the arts afford. This last statement may come as a shock to people who are used to the conventional concept of the true arts and mentally contrast these with mathematics to the detriment of the latter. But the average person has not thought through what the arts really are and what they offer. All that many people actually see in painting, for example, are familiar scenes and perhaps bright colors. These qualities, however, are not the ones which make painting an art. The real values must be learned, and a genuine appreciation of art calls for much study.

Nevertheless, we shall not insist on the aesthetic values of mathematics. It may be fairer to rest on the position that just as there are tone-deaf and color-blind people, so may there be some who temperamentally are intolerant of cold argumentation and the seemingly overfine distinctions of mathematics.

To many people, mathematics offers intellectual challenges, and it is well known that such challenges do engross humans. Games such as bridge, crossword puzzles, and magic squares are popular. Perhaps the best evidence is the attraction of puzzles such as the following: A wolf, a goat, and cabbage are to be transported across a river by a man in a boat which can hold only one of these in addition to the man. How can he take them across so that the wolf does not eat the goat or the goat the cabbage? Two husbands and two wives have to cross a river in a boat which can hold only two people. How can they cross so that no woman is in the company of a man unless her husband is also present? Such puzzles go back to Greek and Roman times. The mathematician Tartaglia, who lived in the sixteenth century, tells us that they were after-dinner amusements.

People do respond to intellectual challenges, and once one gets a slight start in mathematics, he encounters these in abundance. In view of the additional values to be derived from the subject, one would expect people to spend time on mathematical problems as opposed to the more superficial, and in some instances cheap, games which lack depth, beauty, and importance. The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, and the beauty which the ageless mountains present to senses tried by the kaleidoscopic rush of events. The appeal offered by the detachment and objectivity of mathematical reasoning is superbly described by Bertrand Russell.

Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world.

The creation and contemplation of mathematics offer such values.

Despite all these arguments for the study of mathematics, the reader may have justifiable doubts. The idea that thinking about numbers and figures leads to deep and powerful conclusions which influence almost all other branches of thought may seem incredible. The study of numbers and geometrical figures may not seem a sufficiently attractive and promising enterprise. Not even the founders of mathematics envisioned the potentialities of the subject.

So we start with some doubts about the worth of our enterprise. We could encourage the reader with the hackneyed maxim, nothing ventured, nothing gained. We could call to his attention the daily testimony to the power of mathematics offered by almost every newspaper and journal. But such appeals are hardly inspiring. Let us proceed on the very weak basis that perhaps those more experienced in what the world has to offer may also have the wisdom to recommend worth-while studies.

Hence, despite St. Augustine, the reader is invited to tempt hell and damnation by engaging in a study of the subject. Certainly he can be assured that the subject is within his grasp and that no special gifts or qualities of mind are needed to learn mathematics. It is even debatable whether the creation of mathematics requires special talents as does the creation of music or great paintings, but certainly the appreciation of what others have done does not demand a “mathematical mind” any more than the appreciation of art requires an “artistic mind.” Moreover, since we shall not draw upon any previously acquired knowledge, even this potential source of trouble will not arise.

Let us review our objectives. We should like to understand what mathematics is, how it functions, what it accomplishes for the world, and what it has to offer in itself. We hope to see that mathematics has content which serves the physical and social scientist, the philosopher, logician, and the artist; content which influences the doctrines of the statesman and the theologian; content which satisfies the curiosity of the man who surveys the heavens and the man who muses on the sweetness of musical sounds; and content which has undeniably, if sometimes imperceptibly, shaped the course of modern history. In brief, we shall try to see that mathematics is an integral part of the modern world, one of the strongest forces shaping its thoughts and actions, and a body of living though inseparably connected with, dependent upon, and in turn valuable to all other branches of our culture. Perhaps we shall also see how by suffusing and influencing all thought it has set the intellectual temper of our times.

#### EXERCISES

1. A wolf, a goat, and a cabbage are to be rowed across a river in a boat holding only one of these three objects besides the oarsman. How should he carry them across so that the goat should not eat the cabbage or the wolf devour the goat?

2. Another hoary teaser is the following: A man goes to a tub of water with two jars, one holding 3 pt and the other 5 pt. How can he bring back exactly 4 pt?

3. Two husbands and two wives have to cross a river in a boat which can hold only two people. How can they cross so that no woman is in the company of a man unless her husband is also present?

##### Recommended Reading

RUSSELL, BERTRAND: “The Study of Mathematics,” an essay in the collection entitled Mysticism and Logic, Longmans, Green and Co., New York, 1925.

WHITEHEAD, ALFRED NORTH: “The Mathematical Curriculum,” an essay in the collection entitled The Aims of Education, The New American Library, New York, 1949.

WHITEHEAD, ALFRED NORTH: Science and the Modern World, Chaps. 2 and 3, Cambridge University Press, Cambridge, 1926.

## CHAPTER 2

A HISTORICAL ORIENTATION

An educated mind is, as it were, composed of all the minds of preceding ages.

LE BOVIER DE FONTENELLE

2–1 INTRODUCTION

Our first objective will be to gain some historical perspective on the subject of mathematics. Although the logical development of mathematics is not markedly different from the historical, there are nevertheless many features of mathematics which are revealed by a glimpse of its history rather than by an examination of concepts, theorems, and proofs. Thus we may learn what the subject now comprises, how the various branches arose, and how the character of the mathematical contributions made by various civilizations was conditioned by these civilizations. This historical survey may also help us to gain some provisional understanding of the nature, extent, and uses of mathematics. Finally, a preview may help us to keep our bearings. In studying a vast subject, one is always faced with the danger of getting lost in details. This is especially true in mathematics, where one must often spend hours and even days in seeking to understand some new concepts or proofs.

2–2 MATHEMATICS IN EARLY CIVILIZATIONS

Aside possibly from astronomy, mathematics is the oldest and most continuously pursued branch of human thought. Moreover, unlike science, philosophy, and social thought, very little of the mathematics that has ever been created has been discarded. Mathematics is also a cumulative development, that is, newer creations are built logically upon older ones, so that one must usually understand older results to master newer ones. These facts recommend that we go back to the very origins of mathematics.

As we examine the early civilizations, one remarkable fact emerges immediately. Though there have been hundreds of civilizations, many with great art, literature, philosophy, religion, and social institutions, very few possessed any mathematics worth talking about. Most of these civilizations hardly got past the stage of being able to count to five or ten.

In some of these early civilizations a few steps in mathematics were taken. In prehistoric times, which means roughly before 4000 B.C., several civilizations at least learned to think about numbers as abstract concepts. That is, they recognized that three sheep and three arrows have something in common, a quantity called three, which can be thought about independently of any physical objects. Each of us in his own schooling goes through this same process of divorcing numbers from physical objects. The appreciation of “number” as an abstract idea is a great, and perhaps the first, step in the founding of mathematics.

Another step was the introduction of arithmetical operations. It is quite an idea to add the numbers representing two collections of objects in order to arrive at the total instead of counting the objects in the combined collections. Similar remarks apply to subtraction, multiplication, and division. The early methods of carrying out these operations were crude and complicated compared with ours, but the ideas and the applications were there.

Only a few ancient civilizations, Egypt, Babylonia, India, and China, possessed what may be called the rudiments of mathematics. The history of mathematics, and indeed the history of Western civilization, begins with what occurred in the first two of these civilizations. The role of India will emerge later, whereas that of China may be ignored because it was not extensive and moreover had no influence on the subsequent development of mathematics.

Our knowledge of the Egyptian and Babylonian civilizations goes back to about 4000 B.C. The Egyptians occupied approximately the same region that now constitutes modern Egypt and had a continuous, stable civilization from ancient times until about 300 B.C. The term “Babylonian” includes a succession of civilizations which occupied the region of modern Iraq. Both of these peoples possessed whole numbers and fractions, a fair amount of arithmetic, some algebra, and a number of simple rules for finding the areas and volumes of geometrical figures. These rules were but the incidental accumulations of experience, much as people learned through experience what foods to eat. Many of the rules were in fact incorrect but good enough for the simple applications made then. For example, the Egyptian rule for finding the area of a circle amounts to using 3.16 times the square of the radius; that is, their value of π was 3.16. This value, though not accurate, was even better than the several values the Babylonians used, one of these being 3, the value found in the Bible.

What did these early civilizations do with their mathematics? If we may judge from problems found in ancient Egyptian papyri and in the clay tablets of the Babylonians, both civilizations used arithmetic and algebra largely in commerce and state administration, to calculate simple and compound interest on loans and mortgages, to apportion profits of business to the owners, to buy and sell merchandise, to fix taxes, and to calculate how many bushels of grain would make a quantity of beer of a specified alcoholic content. Geometrical rules were applied to calculate the areas of fields, the estimated yield of pieces of land, the volumes of structures, and the quantity of bricks or stones needed to erect a temple or pyramid. The ancient Greek historian Herodotus says that because the annual overflow of the Nile wiped out the boundaries of the farmers’ lands, geometry was needed to redetermine the boundaries. In fact, Herodotus speaks of geometry as the gift of the Nile. This bit of history is a partial truth. The redetermination of boundaries was undoubtedly an application, but geometry existed in Egypt long before the date of 1400 B.C. mentioned by Herodotus for its origin. Herodotus would have been more accurate to say that Egypt is a gift of the Nile, for it is true today as it was then that the only fertile land in Egypt is that along the Nile; and this because the river deposits good soil on the land as it overflows.

Applications of geometry, simple and crude as they were, did play a large role in Egypt and Babylonia. Both peoples were great builders. The Egyptian temples, such as those at Karnak and Luxor, and the pyramids still appear to be admirable engineering achievements even in this age of skyscrapers. The Babylonian temples, called ziggurats, also were remarkable pyramidal structures. The Babylonians were, moreover, highly skilled irrigation engineers, who built a system of canals to feed their hot dry lands from the Tigris and Euphrates rivers.

Perhaps a word of caution is necessary with respect to the pyramids. Because these are impressive structures, some writers on Egyptian civilization have jumped to the conclusion that the mathematics used in the building of pyramids must also have been impressive. These writers point out that the horizontal dimensions of any one pyramid are exactly of the same length, the sloping sides all make the same angle with the ground, and the right angles are right. However, not mathematics but care and patience were required to obtain such results. A cabinetmaker need not be a mathematician.

Mathematics in Egypt and Babylonia was also applied to astronomy. Of course, astronomy was pursued in these ancient civilizations for calendar reckoning and, to some extent, for navigation. The motions of the heavenly bodies give us our fundamental standard of time, and their positions at given times enable ships to determine their location and caravans to find their bearings in the deserts. Calendar reckoning is not only a common daily and commercial need, but it fixes religious holidays and planting times. In Egypt it was also needed to predict the flood of the Nile, so that farmers could move property and cattle away beforehand.

It is worthy of note that by observing the motion of the sun, the Egyptians managed to ascertain that the year contains 365 days. There is a conjecture that the priests of Egypt knew that 365image was a more accurate figure but kept the knowledge secret. The Egyptian calendar was taken over much later by the Romans and then passed on to Europe. The Babylonians, by contrast, developed a lunar calendar. Since the duration of the month as measured from new moon to new moon varies from 29 to 30 days, the twelve-month year adopted by the Babylonians did not coincide with the year of the seasons. Hence the Babylonians added extra months, up to a total of seven, in every 19-year cycle. This scheme was also adopted by the Hebrews.

Astronomy served not only the purposes just described, but from ancient times until recently it also served astrology. In ancient Babylonia and Egypt the belief was widespread that the moon, the planets, and the stars directly influenced and even controlled affairs of the state. This doctrine was gradually extended and later included the belief that the health and welfare of the individual were also subject to the will of the heavenly bodies. Hence it seemed reasonable that by studying the motions and relative positions of these bodies man could determine their influences and even predict his future.

When one compares Egyptian and Babylonian accomplishments in mathematics with those of earlier and contemporary civilizations, one can indeed find reason to praise their achievements. But judged by other standards, Egyptian and Babylonian contributions to mathematics were almost insignificant, although these same civilizations reached relatively high levels in religion, art, architecture, metallurgy, chemistry, and astronomy. Compared with the accomplishments of their immediate successors, the Greeks, the mathematics of the Egyptians and Babylonians is the scrawling of children just learning how to write as opposed to great literature. They barely recognized mathematics as a distinct subject. It was a tool in agriculture, commerce, and engineering, no more important than the other tools they used to build pyramids and zig-gurats. Over a period of 4000 years hardly any progress was made in the subject. Moreover, the very essence of mathematics, namely, reasoning to establish the validity of methods and results, was not even envisioned. Experience recommended their procedures and rules, and with this support they were content. Egyptian and Babylonian mathematics is best described as empirical and hardly deserves the appellation mathematics in view of what, since Greek times, we regard as the chief features of the subject. Some flesh and bones of concrete mathematics were there, but the spirit of mathematics was lacking.

The lack of interest in theoretical or systematic knowledge is evident in all activities of these two civilizations. The Egyptians and Babylonians must have noted the paths of the stars, planets, and moon for thousands of years. Their calendars, as well as tables which are extant, testify to the scope and accuracy of these observations. But no Egyptian or Babylonian strove, so far as we know, to encompass all these observations in one major plan or theory of heavenly motions. Nor does one find any other scientific theory or connected body of knowledge.

2–3 THE CLASSICAL GREEK PERIOD

We have seen so far that mathematics, initiated in prehistoric times, struggled for existence for thousands of years. It finally obtained a firm grip on life in the highly congenial atmosphere of Greece. This land was invaded about 1000 B.C. by people whose origins are not known. By about 600 B.C. these people occupied not only Greece proper but many cities in Asia Minor on the Mediterranean coast, islands such as Crete, Rhodes, and Samos, and cities in southern Italy and Sicily. Though all of these areas bred famous men, the chief cultural center during the classical period, which lasted from about 600 B.C. to 300 B.C., was Athens.

Greek culture was not entirely indigenous. The Greeks themselves acknowledge their indebtedness to the Babylonians and especially to the Egyptians. Many Greeks traveled in Egypt and in Asia Minor. Some went there to study. Nevertheless, what the Greeks created differs as much from what they took over from the Egyptians and Babylonians as gold differs from tin. Plato was too modest in his description of the Greek contribution when he said, “Whatever we Greeks receive we improve and perfect.” The Greeks not only made finished products out of the raw materials imported from Egypt and Babylonia, but they created totally new branches of culture. Philosophy, pure and applied sciences, political thought and institutions, historical writings, almost all our literary forms (except fictional prose), and new ideals such as the freedom of the individual are wholly Greek contributions.

The supreme contribution of the Greeks was to call attention to, employ, and emphasize the power of human reason. This recognition of the power of reasoning is the greatest single discovery made by man. Moreover, the Greeks recognized that reason was the distinctive faculty which humans possessed. Aristotle says, “Now what is characteristic of any nature is that which is best for it and gives most joy. Such to man is the life according to reason, since it is that which makes him man.”

It was by the application of reasoning to mathematics that the Greeks completely altered the nature of the subject. In fact, mathematics as we understand the term today is entirely a Greek gift, though in this case we need not heed Virgil’s injunction to fear such benefactions. But how did the Greeks plan to employ reason in mathematics? Whereas the Egyptians and Babylonians were content to pick up scraps of useful information through experience or trial and error, the Greeks abandoned empiricism and undertook a systematic, rational attack on the whole subject. First of all, the Greeks saw clearly that numbers and geometric forms occur everywhere in the heavens and on earth. Hence they decided to concentrate on these important concepts. Moreover, they were explicit about their intention to treat general abstract concepts rather than particular physical realizations. Thus they would consider the ideal circle rather than the boundary of a field or the shape of a wheel. They then observed that certain facts about these concepts are both obvious and basic. It was evident that equal numbers added to or subtracted from equal numbers give equal numbers. It was equally evident that two right angles are necessarily equal and that a circle can be drawn when center and radius are given. Hence they selected some of these obvious facts as a starting point and called them axioms. Their next idea was to apply reasoning, with these facts as premises, and to use only the most reliable methods of reasoning man possesses. If the reasoning were successful, it would produce new knowledge. Also, since they were to reason about general concepts, their conclusions would apply to all objects of which the concepts were representative. Thus if they could prove that the area of a circle is π times the square of the radius, this fact would apply to the area of a circular field, the floor area of a circular temple, and the cross section of a circular tree trunk. Such reasoning about general concepts might not only produce knowledge of hundreds of physical situations in one proof, but there was always the chance that reasoning would produce knowledge which experience might never suggest. All these advantages the Greeks expected to derive from reasoning about general concepts on the basis of evident reliable facts. A neat plan, indeed!

It is perhaps already clear that the Greeks possessed a mentality totally different from that of the Egyptians and Babylonians. They reveal this also in the plans they had for the use of mathematics. The application of arithmetic and algebra to the computation of interest, taxes, or commercial transactions, and of geometry to the computation of the volumes of granaries was as far from their minds as the most distant star. As a matter of fact, their thoughts were on the distant stars. The Greeks found mathematics valuable in many respects, as we shall learn later, but they saw its main value in the aid it rendered to the study of nature; and of all the phenomena of nature, the heavenly bodies attracted them most. Thus, though the Greeks also studied light, sound, and the motions of bodies on the earth, astronomy was their chief scientific interest.

Just what did the Greeks seek in probing nature? They sought no material gain and no power over nature; they sought merely to satisfy their minds. Because they enjoyed reasoning and because nature presented the most imposing challenge to their understanding, the Greeks undertook the purely intellectual study of nature. Thus the Greeks are the founders of science in the true sense.

The Greek conception of nature was perhaps even bolder than their conception of mathematics. Whereas earlier and later civilizations viewed nature as capricious, arbitrary, and terrifying, and succumbed to the belief that magic and rituals would propitiate mysterious and feared forces, the Greeks dared to look nature in the face. They dared to affirm that nature was rationally and indeed mathematically designed, and that man’s reason, chiefly through the aid of mathematics, would fathom that design. The Greek mind rejected traditional doctrines, supernatural causes, superstitions, dogma, authority, and other such trammels on thought and undertook to throw the light of reason on the processes of nature. In seeking to banish the mystery and seeming arbitrariness of nature and in abolishing belief in dreaded forces, the Greeks were pioneers.

For reasons which will become clearer in a later chapter, the Greeks favored geometry. By 300 B.C., Thales, Pythagoras and his followers, Plato’s disciples, notably Eudoxus, and hundreds of other famous men had built up an enormous logical structure, most of which Euclid embodied in his Elements. This is, of course, the geometry we still study in high school. Though they made some contributions to the study of the properties of numbers and to the solution of equations, almost all of their work was in geometric form, and so there was no improvement over the Egyptians and Babylonians in the representation of, and calculation with, numbers or in the symbolism and techniques of algebra. For these contributions the world had to wait many more centuries. But the vast development in geometry exerted an enormous influence in succeeding civilizations and supplied the inspiration for mathematical activity in civilizations which might otherwise never have acquired even the very concept of mathematics.

The Greek accomplishments in mathematics had, in addition, the broader significance of supplying the first impressive evidence of the power of human reason to deduce new truths. In every culture influenced by the Greeks, this example inspired men to apply reason to philosophy, economics, political theory, art, and religion. Even today Euclid is the prime example of the power and accomplishments of reason. Hundreds of generations since Euclid’s days have learned from his geometry what reasoning is and what it can accomplish. Modern man as well as the ancient Greeks learned from the Euclidean document how exact reasoning should proceed, how to acquire facility in it, and how to distinguish correct from false reasoning. Although many people depreciate this value of mathematics, it is interesting nevertheless that when these people seek to offer an excellent example of reasoning, they inevitably turn to mathematics.

This brief discussion of Euclidean geometry may show that the subject is far from being a relic of the dead past. It remains important as a stepping-stone in mathematics proper and as a paradigm of reasoning. With their gift of reason and with their explicit example of the power of reason, the Greeks founded Western civilization.

2–4 THE ALEXANDRIAN GREEK PERIOD

The intellectual life of Greece was altered considerably when Alexander the Great conquered Greece, Egypt, and the Near East. Alexander decided to build a new capital for his vast empire and founded the city in Egypt named after him. The center of the new Greek world became Alexandria instead of Athens. Moreover, Alexander made deliberate efforts to fuse Greek and Near Eastern cultures. Consequently, the civilization centered at Alexandria, though predominantly Greek, was strongly influenced by Egyptian and Babylonian contributions. This Alexandrian Greek civilization lasted from about 300 B.C. to 600 A.D.

The mixture of the theoretical interests of the Greeks and the practical outlook of the Babylonians and Egyptians is clearly evident in the mathematical and scientific work of the Alexandrian Greeks. The purely geometric investigations of the classical Greeks were continued, and two of the most famous Greek mathematicians, Apollonius and Archimedes, pursued their studies during the Alexandrian period. In fact, Euclid also lived in Alexandria, but his writings reflect the achievements of the classical period. For practical applications, which usually require quantitative results, the Alexandrians revived the crude arithmetic and algebra of Egypt and Babylonia and used these empirically founded tools and procedures, along with results derived from exact geometrical studies. There was some progress in algebra, but what was newly created by men such as Nichomachus and Diophantus was still short of even the elementary methods we learn in high school.

The attempt to be quantitative, coupled with the classical Greek love for the mathematical study of nature, stimulated two of the most famous astronomers of all time, Hipparchus and Ptolemy, to calculate the sizes and distances of the heavenly bodies and to build a sound and, for those times, accurate astronomical theory, which is still known as Ptolemaic theory. Hipparchus and Ptolemy also created the chief tool they needed for this purpose, the mathematical subject known as trigonometry.

During the centuries in which the Alexandrian civilization flourished, the Romans grew strong, and by the end of the third century B.C. they were a world power. After conquering Italy, the Romans conquered the Greek mainland and a number of Greek cities scattered about the Mediterranean area. Among these was the famous city of Syracuse in Sicily, where Archimedes spent most of his life, and where he was killed at the age of 75 by a Roman soldier. According to the account given by the noted historian Plutarch, the soldier shouted to Archimedes to surrender, but the latter was so absorbed in studying a mathematical problem that he did not hear the order, whereupon the soldier killed him.

The contrast between Greek and Roman cultures is striking. The Romans have also bequeathed gifts to Western civilization, but in the fields of mathematics and science their influence was negative rather than positive. The Romans were a practical people and even boasted of their practicality. They sought wealth and world power and were willing to undertake great engineering enterprises, such as the building of roads and viaducts, which might help them to expand, control, and administer their empire, but they would spend no time or effort on theoretical studies which might further these activities. As the great philosopher Alfred North Whitehead remarked, “No Roman ever lost his life because he was absorbed in the contemplation of a mathematical diagram.”

Indirectly as well as directly, the Romans brought about the destruction of the Greek civilization at Alexandria, directly by conquering Egypt and indirectly by seeking to suppress Christianity. The adherents to this new religious movement, though persecuted cruelly by the Romans, increased in number while the Roman Empire grew weaker. In 313 A.D. Rome legalized Christianity and, under the Emperor Theodosius (379–395), adopted it as the official religion of the empire. But even before this time, and certainly after it, the Christians began to attack the cultures and civilizations which had opposed them. By pillage and the burning of books, they destroyed all they could reach of ancient learning. Naturally the Greek culture suffered, and many works wiped out in these holocausts are now lost to us forever.

The final destruction of Alexandria in 640 A.D. was the deed of the Arabs. The books of the Greeks were closed, never to be reopened in this region.

2–5 THE HINDUS AND ARABS

The Arabs, who suddenly appeared on the scene of history in the role of destroyers, had been a nomadic people. They were unified under the leadership of the prophet Mohammed and began an attempt to convert the world to Mohammedanism, using the sword as their most decisive argument. They conquered all the land around the Mediterranean Sea. In the Near East they took over Persia and penetrated as far as India. In southern Europe they occupied Spain, southern France, where they were stopped by Charles Martel, southern Italy and Sicily. Only the Byzantine or Eastern Roman Empire was not subdued and remained an isolated center of Greek and Roman learning. In rather surprisingly quick time as the history of nations goes, the Arabs settled down and built a civilization and culture which maintained a high level from about 800 to 1200 A.D. Their chief centers were Bagdad in what is now Iraq, and Cordova in Spain. Realizing that the Greeks had created wonderful works in many fields, the Arabs proceeded to gather up and study what they could still find in the lands they controlled. They translated the works of Aristotle, Euclid, Apollonius, Archimedes, and Ptolemy into Arabic. In fact, Ptolemy’s chief work, whose title in Greek meant “Mathematical Collection,” was called the Almagest (The Greatest Work) by the Arabs and is still known by this name. Incidentally, other Arabic words which are now common mathematical terms are algebra, taken from the title of a book written by Al-Khowarizmi, a ninth-century Arabian mathematician, and algorithm, now meaning a process of calculation, which is a corruption of the man’s name.

Though they showed keen interest in mathematics, optics, astronomy, and medicine, the Arabs contributed little that was original. It is also peculiar that, although they had at least some of the Greek works and could therefore see what mathematics meant, their own contributions, largely in arithmetic and algebra, followed the empirical, concrete approach of the Egyptians and Babylonians. They could on the one hand appreciate and critically review the precise, exact, and abstract mathematics of the Greeks while, on the other, offer methods of solving equations which, though they worked, had no reasoning to support them. During all the centuries in which Greek works were in their possession, the Arabs manfully resisted the lures of exact reasoning in their own contributions.

We are indebted to the Arabs not only for their resuscitation of the Greek works but for picking up some simple but useful ideas from India, their neighbor on the East. The Indians, too, had built up some elementary mathematics comparable in extent and spirit with the Egyptian and Babylonian developments. However, after about 200 A.D., mathematical activity in India became more appreciable, probably as a result of contacts with the Alexandrian Greek civilization. The Hindus made a few contributions of their own, such as the use of special number symbols from 1 to 9, the introduction of 0, and the use of positional notation with base ten, that is, our modern method of writing numbers. They also created negative numbers. These ideas were taken over by the Arabs and incorporated in their mathematical works.

Because of internal dissension the Arab Empire split into two independent parts. The Crusades launched by the Europeans and the inroads made by the Turks further weakened the Arabs, and their empire and culture disintegrated.

2–6 EARLY AND MEDIEVAL EUROPE

Thus far Europe proper has played no role in the history of mathematics. The reason is simple. The Germanic tribes who occupied central Europe and the Gauls of western Europe were barbarians. Among primitive civilizations, theirs were primitive indeed. They had no learning, no art, no science, not even a system of writing.

The barbarians were gradually civilized. While the Romans were still successful in holding the regions now called France, England, southern Germany, and the Balkans, the barbarians were in contact with, and to some extent influenced by, the Romans. When the Roman Empire collapsed, the Church, already a strong organization, took on the task of civilizing and converting the barbarians. Since the Church did not favor Greek learning and since at any rate the illiterate Europeans had first to learn reading and writing, one is not surprised to find that mathematics and science were practically unknown in Europe until about 1100 A.D.

2–7 THE RENAISSANCE

Insofar as the history of mathematics is concerned, the Arabs served as the agents of destiny. Trade with the Arabs and such invasions of the Arab lands as the Crusades acquainted the Europeans, who hitherto possessed only fragments of the Greek works, with the vast stores of Greek learning possessed by the Arabs. The ideas in these works excited the Europeans, and scholars set about acquiring them and translating them into Latin. Through another accident of history another group of Greek works came to Europe. We have already noted that the Eastern Roman or Byzantine Empire had survived the Germanic and the Arab aggrandizements. But in the fifteenth century the Turks captured the Eastern Roman Empire, and Greek scholars carrying precious manuscripts fled the region and went to Europe.

We shall leave for a later chapter a fuller account of how the European world was aroused by the renaissance of the novel and weighty Greek ideas, and of the challenge these ideas posed to the European beliefs and way of life.[\*](#pg21fn1) From the Greeks the Europeans acquired arithmetic, a crude algebra, the vast development of Euclidean geometry, and the trigonometry of Hip-parchus and Ptolemy. Of course, Greek science and philosophy also became known in Europe.

The first major European development in mathematics occurred in the work of the artists. Imbued with the Greek doctrines that man must study himself and the real world, the artists began to paint reality as they actually perceived it instead of interpreting religious themes in symbolic styles. They applied Euclidean geometry to create a new system of perspective which permitted them to paint realistically. Specifically, the artists created a new style of painting which enabled them to present on canvas, scenes making the same impression on the eye as the actual scenes themselves. From the work of the artists, the mathematicians derived ideas and problems that led to a new branch of mathematics, projective geometry.

Stimulated by Greek astronomical ideas, supplied with data and the astronomical theory of Hipparchus and Ptolemy, and steeped in the Greek doctrine that the world is mathematically designed, Nicolaus Copernicus sought to show that God had done a better job than Hipparchus and Ptolemy had described. The result of Copernicus’ thinking was a new system of astronomy in which the sun was immobile and the planets revolved around the sun. This heliocentric theory was considerably improved by Kepler. Its effects on religion, philosophy, science, and on man’s estimations of his own importance were profound. The heliocentric theory also raised scientific and mathematical problems which were a direct incentive to new mathematical developments.

Just how much mathematical activity the revival of Greek works might have stimulated cannot be determined, for simultaneously with the translation and absorption of these works, a number of other revolutionary developments altered the social, economic, religious, and intellectual life of Europe. The introduction of gunpowder was followed by the use of muskets and later cannons. These inventions revolutionized methods of warfare and gave the newly emerging social class of free common men an important role in that domain. The compass became known to the Europeans and made possible long-range navigation, which the merchants sponsored for the purpose of finding new sources of raw materials and better trade routes. One result was the discovery of America and the consequent influx of new ideas into Europe. The invention of printing and of paper made of rags afforded books in large quantities and at cheap prices, so that learning spread far more than it ever had in any earlier civilizations. The Protestant Revolution stirred debate and doubts concerning doctrines that had been unchallenged for 1500 years. The rise of a merchant class and of free men engaged in labor in their own behalf stimulated an interest in materials, methods of production, and new commodities. All of these needs and influences challenged the Europeans to build a new culture.

2–8 DEVELOPMENTS FROM 1550 TO 1800

Since many of the problems raised by the motion of cannon balls, navigation, and industry called for quantitative knowledge, arithmetic and algebra became centers of attention. A remarkable improvement in these mathematical fields followed. This is the period in which algebra was built as a branch of mathematics and in which much of the algebra we learn in high school was created. Almost all the great mathematicians of the sixteenth and seventeenth centuries, Cardan, Tartaglia, Vieta, Descartes, Fermat, and Newton, men we shall get to know better later, contributed to the subject. In particular, the use of letters to represent a class of numbers, a device which gives algebra its generality and power, was introduced by Vieta. In this same period, logarithms were created to facilitate the calculations of astronomers. The history of arithmetic and algebra illustrates one of the striking and curious features of the history of mathematics. Ideas that seem remarkably simple once explained were thousands of years in the making.

The next development of consequence, coordinate geometry, came from two men, both interested in method. One was René Descartes. Descartes is perhaps even more famous as a philosopher than as a mathematician, though he was one of the major contributors to our subject. As a youth Descartes was already troubled by the intellectual turmoil of his age. He found no certainty in any of the knowledge taught him, and he therefore concentrated for years on finding the method by which man can arrive at truths. He found the clue to this method in mathematics, and with it constructed the first great modern philosophical system. Because the scientific problems of his time involved work with curves, the paths of ships at sea, of the planets, of objects in motion near the earth, of light, and of projectiles, Descartes sought a better method of proving theorems about curves. He found the answer in the use of algebra. Pierre de Fermat’s interest in method was confined to mathematics proper, but he too appreciated the need for more effective ways of working with curves and also arrived at the idea of applying algebra. In this development of coordinate geometry we have one of the remarkable examples of how the times influence the direction of men’s thoughts.

We have already noted that a new society was developing in Europe. Among its features were expanded commerce, manufacturing, mining, large-scale agriculture, and a new social class—free men working as laborers or as independent artisans. These activities and interests created problems of materials, methods of production, quality of the product, and utilization of devices to replace or increase the effectiveness of manpower. The people involved, like the artists, had become aware of Greek mathematics and science and sensed that it could be helpful. And so they too sought to employ this knowledge in their own behalf. Thereby arose a new motive for the study of mathematics and science. Whereas the Greeks had been content to study nature merely to satisfy their own curiosity and to organize their conclusions in patterns pleasing to the mind, the new goal, effectively proclaimed by Descartes and Francis Bacon, was to make nature serve man. Hence mathematicians and scientists turned earnestly to an enlarged program in which both understanding and mastery of nature were to be sought.

However, Bacon had cautioned that nature can be commanded only when one learns to obey her. One must have facts of nature on which to base reasoning about nature. Hence mathematicians and scientists sought to acquire facts from the experience of artists, technicians, artisans, and engineers. The alliance of mathematics and experience was gradually transformed into an alliance of mathematics and experimentation, and a new method for the pursuit of the truths of nature, first clearly perceived and formulated by Galileo Galilei (1564–1642) and Newton, was gradually evolved. The plan, perhaps oversimply stated, was that experience and experiment were to supply basic mathematical principles and mathematics was to be applied to these principles to deduce new truths, just as new truths are deduced from the axioms of geometry.

The most pressing scientific problem of the seventeenth century was the study of motion. On the practical side, investigations of the motion of projectiles, of the motion of the moon and planets to aid navigation, and of the motion of light to improve the design of the newly discovered telescope and microscope, were the primary interests. On the theoretical side, the new heliocentric astronomy invalidated the older, Aristotelian laws of motion and called for totally new principles. It was one thing to explain why a ball fell to earth on the assumption that the earth was immobile and the center of the universe, and another to explain this phenomenon in the light of the fact that the earth was rotating and revolving around the sun. A new science of motion was created by Galileo and Newton, and in the process two brand-new developments were added to mathematics. The first of these was the notion of a function, a relationship between variables best expressed for most purposes as a formula. The second, which rests on the notion of a function but represents the greatest advance in method and content since Euclid’s days, was the calculus. The subject matter of mathematics and the power of mathematics expanded so greatly that at the end of the seventeenth century Leibniz could say,

Taking mathematics from the beginning of the world to the time when Newton lived, what he had done was much the better half.

With the aid of the calculus Newton was able to organize all data on earthly and heavenly motions into one system of mathematical mechanics which encompassed the motion of a ball falling to earth and the motion of the planets and stars. This great creation produced universal laws which not only united heaven and earth but revealed a design in the universe far more impressive than man had ever conceived. Galileo’s and Newton’s plan of applying mathematics to sound physical principles not only succeeded in one major area but gave promise, in a rapidly accelerating scientific movement, of embracing all other physical phenomena.

We learn in history that the end of the seventeenth century and the eighteenth century were marked by a new intellectual attitude briefly described as the Age of Reason. We are rarely told that this age was inspired by the successes which mathematics, to be sure in conjunction with science, had achieved in organizing man’s knowledge. Infused with the conviction that reason, personified by mathematics, would not only conquer the physical world but could solve all of man’s problems and should therefore be employed in every intellectual and artistic enterprise, the great minds of the age undertook a sweeping reorganization of philosophy, religion, ethics, literature, and aesthetics. The beginnings of new sciences such as psychology, economics, and politics were made during these rational investigations. Our principal intellectual doctrines and outlook were fashioned then, and we still live in the shadow of the Age of Reason.

While these major branches of our culture were being transformed, eighteenth-century scientists continued to win victories over nature. The calculus was soon extended to a new branch of mathematics called differential equations, and this new tool enabled scientists to tackle more complex problems in astronomy, in the study of the action of forces causing motions, in sound, especially musical sounds, in light, in heat (especially as applied to the development of the steam engine), in the strength of materials, and in the flow of liquids and gases. Other branches, which can be merely mentioned, such as infinite series, the calculus of variations, and differential geometry, added to the extent and power of mathematics. The great names of the Bernoullis, Euler, Lagrange, Laplace, d’Alembert, and Legendre, belong to this period.

2–9 DEVELOPMENTS FROM 1800 TO THE PRESENT

During the nineteenth century, developments in mathematics came at an ever increasing rate. Algebra, geometry and analysis, the last comprising those subjects which stem from calculus, all acquired new branches. The great mathematicians of the century were so numerous that it is impractical to list them. We shall encounter some of the greatest of these, Karl Friedrich Gauss and Bernhard Riemann, in our work. We might mention also Henri Poincaré and David Hilbert, whose work extended into the twentieth century.

Undoubtedly the primary cause of this expansion in mathematics was the expansion in science. The progress made in the seventeenth and eighteenth centuries had sufficiently illustrated the effectiveness of science in penetrating the mysteries of the physical world and in giving man control over nature, to cause an all the more vigorous pursuit of science in the nineteenth century. In that century also, science became far more intimately linked with engineering and technology than ever before. Mathematicians, working closely with the scientists as they had since the seventeenth century, were presented with thousands of significant physical problems and responded to these challenges.

Perhaps the major scientific development of the century, which is typical in its stimulation of mathematical activity, was the study of electricity and magnetism. While still in its infancy this science yielded the electric motor, the electric generator, and telegraphy. Basic physical principles were soon expressed mathematically, and it became possible to apply mathematical techniques to these principles, to deduce new information just as Galileo and Newton had done with the principles of motion. In the course of such mathematical investigations, James Clerk Maxwell discovered electromagnetic waves of which the best known representatives are radio waves. A new world of phenomena was thus uncovered, all embraced in one mathematical system. Practical applications, with radio and television as most familiar examples, soon followed.

Remarkable and revolutionary developments of another kind also took place in the nineteenth century, and these resulted from a re-examination of elementary mathematics. The most profound in its intellectual significance was the creation of non-Euclidean geometry by Gauss. His discovery had both tantalizing and disturbing implications: tantalizing in that this new field contained entirely new geometries based on axioms which differ from Euclid’s, and disturbing in that it shattered man’s firmest conviction, namely that mathematics is a body of truths. With the truth of mathematics undermined, realms of philosophy, science, and even some religious beliefs went up in smoke. So shocking were the implications that even mathematicians refused to take non-Euclidean geometry seriously until the theory of relativity forced them to face the full significance of the creation.

For reasons which we trust will become clearer further on, the devastation caused by non-Euclidean geometry did not shatter mathematics but released it from bondage to the physical world. The lesson learned from the history of non-Euclidean geometry was that though mathematicians may start with axioms that seem to have little to do with the observable behavior of nature, the axioms and theorems may nevertheless prove applicable. Hence mathematicians felt freer to give reign to their imaginations and to consider abstract concepts such as complex numbers, tensors, matrices, and n-dimensional spaces. This development was followed by an even greater advance in mathematics and, surprisingly, an increasing use of mathematics in the sciences.

Even before the nineteenth century, the rationalistic spirit engendered by the success of mathematics in the study of nature penetrated to the social scientists. They began to emulate the physical scientists, that is, to search for the basic truths in their fields and to attempt reorganization of their subjects on the mathematical pattern. But these attempts to deduce the laws of man and society and to erect sciences of biology, economics, and politics did not succeed, although they did have some indirect beneficial effects.

The failure to penetrate social and biological problems by the deductive method, that is, the method of reasoning from axioms, caused social scientists to take over and develop further the mathematical theories of statistics and probability, which had already been initiated by mathematicians for various purposes ranging from problems of gambling to the theory of heat and astronomy. These techniques have been remarkably successful and have given some scientific methodology to what were largely speculative domains.

This brief sketch of the mathematics which will fall within our purview may make it clear that mathematics is not a closed book written in Greek times. It is rather a living plant that has flourished and languished with the rise and fall of civilizations. Since about 1600 it has been a continuing development which has become steadily vaster, richer, and more profound. The character of mathematics has been aptly, if somewhat floridly, described by the nineteenth-century English mathematician James Joseph Sylvester.

Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possession, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined; it is as limitless as the space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer’s gaze; it is incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity as the consciousness, the life, which seems to slumber in each monad, in every atom of matter, in each leaf and bud and cell and is forever ready to burst forth into new forms of vegetable and animal existence.

Our sketch of the development of mathematics has attempted to indicate the major eras and civilizations in which the subject has flourished, the variety of interests which induced people to pursue mathematics, and the branches of mathematics that have been created. Of course, we intend to investigate more carefully and more fully what these creations are and what values they have furnished to mankind. One fact of history may be noted by way of summary here. Mathematics as a body of reasoning from axioms stems from one source, the classical Greeks. All other civilizations which have pursued or are pursuing mathematics acquired this concept of mathematics from the Greeks. The Arab and Western European were the next civilizations to take over and expand on the Greek foundation. Today countries such as the United States, Russia, China, India, and Japan are also active. Though the last three of these did possess some native mathematics, it was limited and empirical as in Babylonia and Egypt. Modern mathematical activity in these five countries and wherever else it is now taking hold was inspired by Western European thought and actually learned by men who studied in Europe and returned to build centers of teaching in their own countries.

2–10 THE HUMAN ASPECT OF MATHEMATICS

One final point about mathematics is implicit in what we have said. We have spoken of problems which gave rise to mathematics, of cultures which emphasized some directions of thinking as opposed to others, and of branches of mathematics, as though all these forces and activities were as impersonal as the force of gravitation. But ideas and thinking are conveyed by people. Mathematics is a human creation. Although most Greeks did believe that mathematics existed independently of human beings as the planets and mountains seem to, and that all that human beings do is discover more and more of the structure, the prevalent belief today is that mathematics is entirely a human product. The concepts, the axioms, and the theorems established are all created by human beings in man’s attempt to understand his environment, to give play to his artistic instincts, and to engage in absorbing intellectual activity.

The lives and activities of the men themselves are also fascinating. While mathematicians produce formulas, no formula produces mathematicians. They have come from all levels of society. The special talent, if there is such, which makes mathematicians has been found in Casanovas and ascetics, among business men and philosophers, among atheists and the profoundly religious, among the retiring and the worldly. Some, like Blaise Pascal and Gauss, were precocious; Évariste Galois was dead at 21, and Niels Hendrik Abel at 27. Others, like Karl Weierstrass and Henri Poincaré, matured more normally and were productive throughout their lives. Many were modest; others extremely egotistical and vain beyond toleration. One finds scoundrels, such as Cardan, and models of rectitude. Some were generous in their recognition of other great minds; others were resentful and jealous and even stole ideas to boost their own reputations. Disputes about priority of discovery abound.

The point in learning about these human variations, aside from satisfying our instinct to pry into other people’s lives, is that it explains to a large extent why the progress of the highly rational subject of mathematics has been highly irrational. Of course, the major historical forces, which we sketched above, limit the actions and influence the outlook of individuals, but we also find in the history of mathematics all the vagaries which he have learned to associate with human beings. Leading mathematicians have failed to recognize bright ideas suggested by younger men, and the authors died neglected. Big men and little men made unsuccessful attempts to solve problems which their successors solved with ease. On the other hand, some supposed proofs offered even by masters were later found to be false. Generations and even ages failed to note new ideas, despite the fact that all that was needed was not a technical achievement but merely a point of view. The examples of the blindness of human beings to ideas which later seem simple and obvious furnish fascinating insight into the working of the human mind.

Recognition of the human element in mathematics explains in large measure the differences in the mathematics produced by different civilizations and the sudden spurts made in new directions by virtue of insights supplied by genius. Though no subject has profited as much as mathematics has by the cumulative effect of thousands of workers and results, in no subject is the role of great minds more readily discernible.

#### EXERCISES

1. Name a few civilizations which contributed to mathematics.

2. What basis did the Egyptians and Babylonians have for believing in their mathematical methods and formulas?

3. Compare Greek and pre-Greek understanding of the concepts of mathematics.

4. What was the Greek plan for establishing mathematical conclusions?

5. What was the chief contribution of the Arabs to the development of mathematics?

6. In what sense is mathematics a creation of the Greeks rather than of the Egyptians and Babylonians?

7. Criticize the statement “Mathematics was created by the Greeks and very little was added since their time.”

##### Topics for Further Investigation

To write on the following topics use the books listed under Recommended Reading.

1. The mathematical contributions of the Egyptians or Babylonians.

2. The mathematical contributions of the Greeks.

##### Recommended Reading

BALL, W. W. ROUSE: A Short Account of the History of Mathematics, Dover Publications, Inc., New York, 1960.

BELL, ERIC T.: Men of Mathematics, Simon and Schuster, New York, 1937.

CHILDE, V. GORDON: Man Makes Himself, The New American Library, New York, 1951.

EVES, HOWARD: An Introduction to the History of Mathematics, Rev. ed., Holt, Rinehart and Winston, Inc., New York, 1964.

NEUGEBAUER, OTTO: The Exact Sciences in Antiquity, Princeton University Press, Princeton, 1952.

SCOTT, J. F.: A History of Mathematics, Taylor and Francis, Ltd., London, 1958.

SMITH, DAVID EUGENE: History of Mathematics, Vol. I, Dover Publications, Inc., New York, 1958.

STRUIK, DIRK J.: A Concise History of Mathematics, Dover Publications, Inc., New York, 1948.

[\*](#ipg21fn1) See [Chapter 9](#ch9).

## CHAPTER 3

LOGIC AND MATHEMATICS

Geometry will draw the soul toward truth and create the spirit of philosophy.

PLATO

3–1 INTRODUCTION

Mathematics has its own ways of establishing knowledge, and the understanding of mathematics is considerably promoted if one learns first just what those ways are. In this chapter we shall study the concepts which mathematics treats; the method, called deductive proof, by which mathematics establishes its conclusions; and the principles or axioms on which mathematics rests. Study of the contents and logical structure of mathematics leaves untouched the subject of how the mathematician knows what conclusions to establish and how to prove them. We shall therefore present a brief and preliminary discussion of the creation of mathematics. This topic will recur as we examine the subject matter itself in subsequent chapters.

Since mathematics, as we conceive the subject today, was fashioned by the Greeks, we shall also attempt to see what features of Greek thought and culture caused these people to remodel what the Egyptians and Babylonians had pursued for several thousand years.

3–2 THE CONCEPTS OF MATHEMATICS

The first major step which the Greeks made was to insist that mathematics must deal with abstract concepts. Let us see just what this means. When we first learn about numbers we are taught to think about collections of particular objects such as two apples, three men, and so on. Gradually and rather subconsciously we begin to think about the numbers 2, 3, and other whole numbers without having to associate them with physical objects. We soon reach the more advanced stage of adding, subtracting, and performing other operations with numbers without having to handle collections of objects in order to understand these operations or to see that the results agree with experience. Thus we soon become convinced that 4 times 5 must be 20, whether these numbers represent quantities of apples, horses, or even purely imaginary objects. By this time we are really dealing with concepts or ideas, for the whole numbers do not exist in nature. Any whole number is rather an abstraction of a property which is common to many different collections or sets of objects.

The whole numbers then are ideas, and the same is true of fractions such as image, image, and so on. In the latter case, too, the formulation of the physical relationship of a part of an object to the whole, whether it refers to pies, bushels of wheat, or to a smaller monetary value in relation to a larger one, again leads to an abstraction. Mathematicians formulate operations with fractions, that is, combining parts of an object, taking one part away from the other, or taking a part of a part, in such a way that the result of any operation on abstract fractions agrees with the corresponding physical occurrence. Thus the mathematical process of, say adding image and image, which yields image, expresses the addition of image of a pie and image of a pie, and the result tells us how many parts of a pie one would actually have.

Whole numbers, fractions, and the various operations with whole numbers and fractions are abstractions. Although this fact is rather easy to understand, we tend to lose sight of it and cause ourselves unnecessary confusion. Let us consider an example. A man goes into a shoe store and buys 3 pairs of shoes at 10 dollars per pair. The storekeeper reasons that 3 pairs times 10 dollars is 30 dollars and asks for 30 dollars in return for the 3 pairs of shoes. If this reasoning is correct, then it is equally correct for the customer to argue that 3 pairs times 10 dollars is 30 pairs of shoes and to walk out with 30 pairs of shoes without handing the storekeeper one cent. The customer may end up in jail, but he may console himself while he languishes there that his reasoning is as sound as the storekeeper’s.

The source of the difficulty is, of course, that one cannot multiply shoes by dollars. One can multiply the number 3 by the number 10 and obtain the number 30. The practical and no doubt obligatory physical interpretation of the answer in the above situation is that one must pay 30 dollars rather than walk out with 30 pairs of shoes. We see, therefore, that one must distinguish between the purely mathematical operation of multiplying 3 by 10 and the physical objects with which these numbers may be associated.

The same point is involved in a slightly different situation. Mathematically image is equal to image. But the corresponding physical fact may not be true. One may be willing to accept 4 half-pies instead of 2 whole pies, but no woman would accept 4 half-dresses in place of 2 dresses or 4 half-shoes in place of 1 pair of whole shoes.

The Egyptians and Babylonians did reach the stage of working with pure numbers dissociated from physical objects. But like young children of our civilization, they hardly recognized that they were dealing with abstract entities. By contrast, the Greeks not only recognized numbers as ideas but emphasized that this is the way we must regard them. The Greek philosopher Plato, who lived from 428 to 348 B.C. and whose ideas are representative of the classical Greek period, says in his famous work, the Republic,

We must endeavor that those who are to be the principal men of our State go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; . . . arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument.

The Greeks not only emphasized the distinction between pure numbers and the physical applications of such numbers, but they preferred the former to the latter. The study of the properties of pure numbers, which they called arithmetica, was esteemed as a worthy activity of the mind, whereas the use of numbers in practical applications, logistica, was deprecated as a mere skill.

Geometrical thinking prior to the classical Greek period was even less advanced than thinking about numbers. To the Egyptians and Babylonians the words “straight line” meant no more than a stretched rope or a line traced in sand, and a rectangle was a piece of land of a particular shape. The Greeks began the practice of treating point, line, triangle, and other geometrical notions as concepts. They did of course appreciate that these mental notions are suggested by physical objects, but they stressed that the concepts differ from the physical examples as sharply as the concept of time differs from the passage of the sun across the sky. The stretched string is a physical object illustrating the concept of line, but the mathematical line has no thickness, no color, no molecular structure, and no tension.

The Greeks were explicit in asserting that geometry deals with abstractions. Speaking of mathematicians, Plato says,

And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter . . . they are really seeking to behold the things themselves, which can be seen only with the eye of the mind?

On the basis of elementary abstractions, mathematics creates others which are even more remote from anything real. Negative numbers, equations involving unknowns, formulas, and other concepts we shall encounter are abstractions built upon abstractions. Fortunately, every abstraction is ultimately derived from, and therefore understandable in terms of, intuitively meaningful objects or phenomena. The mind does play its part in the creation of mathematical concepts, but the mind does not function independently of the outside world. Indeed the mathematician who treats concepts that have no physically real or intuitive origins is almost surely talking nonsense. The intimate connection between mathematics and objects and events in the physical world is reassuring, for it means that we can not only hope to understand the mathematics proper, but also expect physically meaningful and valuable conclusions.

The use of abstractions is not peculiar to mathematics. The concepts of force, mass, and energy, which are studied in physics, are abstractions from real phenomena. The concept of wealth, an abstraction from material possessions such as land, buildings, and jewelry, is studied in economics. The concepts of liberty, justice, and democracy are familiar in political science. Indeed, with respect to the use of abstract concepts, the distinction between mathematics on the one hand and the physical and social sciences on the other is not a sharp one. In fact, the influence of mathematics and mathematical ways of thinking on the physical sciences especially has led to ever increasing use of abstract concepts including some, as we shall see, which may have no direct real counterpart at all, any more than a mathematical formula has a direct real counterpart.

The very fact that other studies also engage in abstractions raises an important question. Mathematics is confined to some abstractions, numbers and geometrical forms, and to concepts built upon these basic ones. Abstractions such as mass, force, and energy belong to physics, and still other abstractions belong to other subjects. Why doesn’t mathematics also treat forces, wealth, and justice? Certainly these concepts are also worthy of study. Did the mathematicians make an agreement with physicists, economists, and others to divide the concepts among themselves? The restriction of mathematics to numbers and geometrical forms is partly a historical accident and partly a deliberate decision made by the Greeks. Numbers and geometrical forms had already been introduced by the Egyptians and Babylonians, and their utility in daily life was established. Since the Greeks learned the rudiments of mathematics from these civilizations, the sheer weight of tradition might have caused them to continue the practice of regarding mathematics as the study of numbers and geometrical figures. But people as original and bold in thought as the Greeks would not have been bound merely by tradition, had they not found in numbers and geometrical forms sharp and clear notions which appealed to their delight in the processes of exact thinking. However, an even more compelling reason was their belief that numerical and geometrical properties and relationships were basic, that they underlay the phenomena of the physical world and the design of the entire universe. Hence to understand the world one should seek this mathematical essence. The brilliance and depth of their conception of the universe will be revealed more and more as we proceed.

When one compares the pre-Greek and Greek understanding of the concepts of mathematics and notes the sharp transition from the concrete to the abstract, another question presents itself. The Greeks eliminated the physical substance and retained only the idea. Why did they do it? Surely it is more difficult to think about abstractions than about concrete things. Also it would seem that an attempt to study nature by concentrating on just a few aspects of physical objects rather than on the objects themselves would fall far short of effectiveness.

Insofar as the emphasis on abstractions is concerned, the Greeks saw at once what any thinking people would see sooner or later. One advantage of treating abstractions is the gain in generality. When a child learns that 5 + 5 = 10, he acquires in one swoop a fact which applies to hundreds of situations. Likewise a theorem proved about the abstract triangle applies to a triangular piece of land, a musical percussion instrument, and a triangle determined by three heavenly bodies at any instant of time. It has been said that the process of abstraction amounts to giving the same name to different things, but this very recognition that different objects possess the common property named in the abstraction carries with it the implication that anything true of the abstraction will apply to the several objects. Part of the secret of the power of mathematics is that it deals with abstractions.

Another advantage of abstraction was also clear to the Greeks. Abstracting from a physical situation just those properties which are to be studied frees the mind from burdensome and irrelevant details and enables one to concentrate on the features of interest. When one wishes to determine the area of a piece of land, only shape and size are relevant, and it is desirable to think only about these and not about the fertility of the soil.

The emphasis on mathematical abstractions by the classical Greeks was part and parcel of their outlook on the entire universe. They were concerned with truths, and leading philosophical schools, notably the Pythagoreans and the Platonists, maintained that truths could be established only about abstractions. Let us follow their argument. The physical world presents various objects to the senses. But the impressions received by the senses are inexact, transitory, and constantly changing; indeed, the senses may be even deceived, as by mirages. However, truth, by its very meaning, must consist of permanent, unchanging, definite entities and relationships. Fortunately, the intelligence of man excited to reflection by the impressions of sensible objects may rise to higher conceptions of the realities faintly exhibited to the senses, and so man may rise to the contemplation of ideas. These are eternal realities and the true goal of thought, whereas mere “things are the shadows of ideas thrown on the screen of experience.”

Thus Plato would say that there is nothing real in a horse, a house, or a beautiful woman. The reality is in the universal type or idea of a horse, a home, or a woman. The ideas, among which Plato emphasized Beauty, Justice, Intelligence, Goodness, Perfection, and the State, are independent of the superficial appearances of things, of the flux of life, and of the biases and warped desires of man; they are in fact constant and invariable, and knowledge concerning them is firm and indestructible. Real and eternal knowledge concerns these ideas, rather than sensuous objects. This distinction between the intelligible world and the world revealed by the senses is all-important in Plato.



Fig. 3–1.  
Polyclitus: Spear-bearer (Daryphorus). National Museum, Naples.

To put Plato’s doctrine in everyday language, fundamental knowledge does not concern itself with what John ate, Mary heard, or William felt. Knowledge must rise above individuals and particular objects and tell us about broad classes of objects and about man as a whole. True knowledge must therefore of necessity concern abstractions. Plato admits that physical or sensible objects suggest the ideas just as diagrams of geometry suggest abstract geometrical concepts. Hence there is a point to studying physical objects, but one must not lose himself in trivial and confusing minutiae.

The abstractions of mathematics possessed a special importance for the Greeks. The philosophers pointed out that, to pass from a knowledge of the world of matter to the world of ideas, man must train his mind to grasp the ideas. These highest realities blind the person who is not prepared to contemplate them. He is, to use Plato’s famous simile, like one who lives continuously in the deep shadows of a cave and is suddenly brought out into the sunlight. The study of mathematics helps make the transition from darkness to light. Mathematics is in fact ideally suited to prepare the mind for higher forms of thought because on the one hand it pertains to the world of visible things and on the other hand it deals with abstract concepts. Hence through the study of mathematics man learns to pass from concrete figures to abstract forms; moreover, this study purifies the mind by drawing it away from the contemplation of the sensible and perishable and leading it to the eternal ideas. These latter abstractions are on the same mental level as the concepts of mathematics. Thus, Socrates says, “The understanding of mathematics is necessary for a sound grasp of ethics.”

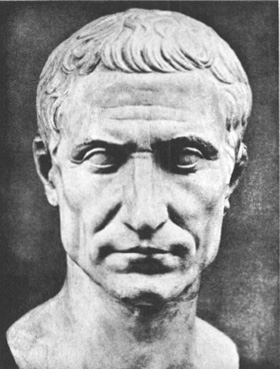


Fig. 3–2.  
Bust of Caesar. Vatican.

To sum up Plato’s position we may say that while a little knowledge of geometry and calculation suffices for practical needs, the higher and more advanced portions tend to lift the mind above mundane considerations and enable it to apprehend the final aim of philosophy, the idea of the Good. Mathematics, then, is the best preparation for philosophy. For this reason Plato recommended that the future rulers, who were to be philosopher-kings, be trained for ten years, from age 20 to 30, in the study of the exact sciences, arithmetic, plane geometry, solid geometry, astronomy, and harmonics (music). The oft-repeated inscription over the doors of Plato’s Academy, stating that no one ignorant of mathematics should enter, fully expresses the importance he attached to the subject, although modern critics of Plato read into these words his admission that one would not be able to learn it after entering. This value of mathematical training led one historian to remark, “Mathematics considered as a science owes its origins to the idealistic needs of the Greek philosophers, and not as fable has it, to the practical demands of Egyptian economics.”



Fig. 3–3.  
Parthenon, Athens.

The preference of the Greeks for abstractions is equally evident in the art of the great sculptors, Polyclitus, Praxiteles, and Phidias. One has only to glance at the face in [Fig. 3–1](#fig3_1) to observe that Greek sculpture of the classical period dwelt not on particular men and women but on types, ideal types. Idealization extended to standardization of the ratios of the parts of the body to each other. Polyclitus believed, in fact, that there were ideal numerical ratios which fix the proportions of the human body. Perfect art must follow these ideal proportions. He wrote a book, The Canon, on the subject and constructed the “Spear-bearer” to illustrate the thesis. These abstract types contrast sharply with what is found in numerous busts and statues of private individuals and military and political leaders made by Romans ([Fig. 3–2](#fig3_2)).

Greek architecture also reveals the emphasis on ideal forms. The simple and austere buildings were always rectangular in shape; even the ratios of the dimensions employed were fixed. The Parthenon at Athens ([Fig. 3–3](#fig3_3)) is an example of the style and proportions found in almost all Greek temples.

#### EXERCISES

1. Suppose 5 trucks pass by with 4 men in each. To answer the question of how many men there are in all the trucks, a person reasons that 4 men times 5 trucks is 20 men. On the other hand, if there are 4 men each owning 5 trucks, the total number of trucks is 20 trucks. Hence 4 men times 5 trucks yields 20 trucks. How do you know that the answer is 20 men in one case and 20 trucks in the other?

2. If the product of 25¢ and 25¢ is obtained by multiplying 0.25 by 0.25 the result is 0.0625 or 6 image¢. Does it pay to multiply money?

3. Can you suggest some abstract political or ethical concepts?

4. Suppose 30 books are to be distributed among 5 people. Since 30 books divided by 5 people yields 6 books, each person gets 6 books. Criticize the reasoning.

5. A store advertises that it will give a credit of $1 for each purchase amounting to $1. A man who spends $6 reasons that he should receive a credit of $6 times $1, or $6. But $6 is 600¢ and $1 is 100¢. Hence 600¢ times 100¢ is 60,000¢, or $600. It would seem that it is more profitable to operate with the almost worthless cent than with dollars! What is wrong?

6. What does the statement that mathematics deals with abstractions mean?

7. Why did the Greeks make mathematics abstract?

3–3 IDEALIZATION

The geometrical notions of mathematics are abstract in the sense that shapes are mental concepts which actual physical objects merely approximate. The sides of a rectangular piece of land may not be exactly straight nor would each angle be exactly 90°. Hence, in adopting such abstract concepts, mathematics does idealize. But in studying the physical world, mathematics also idealizes in another sense which is equally important. Very often mathematicians undertake to study an object which is not a sphere and yet choose to regard it as such. For example, the earth is not a sphere but a spheroid, that is, a sphere flattened at the top and bottom. Yet in many physical problems which are treated mathematically the earth is represented as a perfect sphere. In problems of astronomy a large mass such as the earth or the sun is often regarded as concentrated at one point.

In making such idealizations, the mathematician deliberately distorts or approximates at least some features of the physical situation. Why does he do it? The reason usually is that he simplifies the problem and yet is quite sure that he has not introduced any gross errors. If one is to investigate, for example, the motion of a shell which travels ten miles, the difference between the assumed spherical shape of the earth and the true spheroidal shape does not matter. In fact, in the study of any motion which takes place over a limited region, say one mile, it may be sufficient to treat the earth as a flat surface. On the other hand, if one were to draw a very accurate map of the earth, he would take into account that the shape is spheroidal. As another example, to find the distance to the moon, it is good enough to assume that the moon is a point in space. However, to find the size of the moon, it is clearly pointless to regard the moon as a point.

The question does arise, how does the mathematician know when idealization is justified? There is no simple answer to this question. If he has to solve a series of like problems, he may solve one using the correct figure, and another, using a simplified figure, and compare results. If the difference does not matter for his purposes, he may then retain the simpler figure for the remaining problems. Sometimes he can estimate the error introduced by using the simpler figure and may find that this error is too small to matter. Or the mathematician may make the idealization and use the result because it is the best he can do. Then he must accept experience as his guide in deciding whether the result is good enough.

To idealize by deliberately introducing a simplification is to lie a little, but the lie is a white one. Using idealizations to study the physical world does impose a limitation on what mathematics accomplishes, but we shall find that even where idealizations are employed, the knowledge gained is of immense value.

#### EXERCISES

1. Distinguish between abstraction and idealization.

2. Is it correct to assume that the lines of sight to the sun from two places A and B on the earth’s surface are parallel?

3. Suppose you wished to measure the height of a flagpole. Would it be wise to regard the flagpole as a line segment?

3–4 METHODS OF REASONING

There are many ways, more or less reliable, of obtaining knowledge. One can resort to authority as one often does in obtaining historical knowledge. One may accept revelation as many religious people do. And one may rely upon experience. The foods we eat are chosen on the basis of experience. No one determined in advance by careful chemical analysis that bread is a healthful food.

We may pass over with a mere mention such sources of knowledge as authority and revelation, for these sources cannot be helpful in building mathematics or in acquiring knowledge of the physical world. It is true that in the medieval period of Western European culture men did contend that all desirable knowledge of nature was revealed in the Bible. However in no significant period of scientific thought has this view played any role. Experience, on the other hand, is a useful source of knowledge. But there are difficulties in employing this method. We should not wish to build a fifty-story building in order to decide whether a steel beam of specified dimensions is strong enough to be used in the foundation. Moreover, even if one should happen to choose workable dimensions, the choice may be wasteful of materials. Of course, experience is of no use in determining the size of the earth or the distance to the moon.

Closely related to experience is the method of experiment which amounts to setting up and going through a series of purposive, systematic experiences. It is true that experimentation fundamentally is experience, but it is usually accompanied by careful planning which eliminates extraneous factors, and the experience is repeated enough times to yield reliable information. However, experimentation is subject to much the same limitations as experience.

Are authority, revelation, experience, and even experimentation the only methods of obtaining knowledge? The answer is no. The major method is reasoning, and within the domain of reasoning there are several forms. One can reason by analogy. A boy who is considering a college career may note that his friend went to college and handled it successfully. He argues that since he is very much like his friend in physical and mental qualities, he too should succeed in college work. The method of reasoning just illustrated is to find a similar situation or circumstance and to argue that what was true for the similar case should be true of the one in question. Of course, one must be able to find a similar situation and one must take the chance that the differences do not matter.

Another common method of reasoning is induction. People use this method of reasoning every day. Because a person may have had unfortunate experiences in dealing with a few department stores, he concludes that all department stores are bad to deal with. Or, for example, experimentation would show that iron, copper, brass, oil, and other substances expand when heated, and one consequently concludes that all substances expand when heated. Inductive reasoning is in fact the common method used in experimentation. An experiment is generally performed many times, and if the same result is obtained each time, the experimenter concludes that the result will always follow. The essence of induction is that one observes repeated occurrences of the same phenomenon and concludes that the phenomenon will always occur. Conclusions obtained by induction seem well warranted by the evidence, especially when the number of instances observed is large. Thus the sun is observed so often to rise in the morning that one is sure it has risen even on those mornings when it is hidden by clouds.

There is still a third method of reasoning, called deduction. Let us consider some examples. If we accept as basic facts that honest people return found money and that John is honest, we may conclude unquestionably that John will return money that he finds. Likewise, if we start with the facts that no mathematician is a fool and that John is a mathematician, then we may conclude with certainty that John is not a fool. In deductive reasoning we start with certain statements, called premises, and assert a conclusion which is a necessary or inescapable consequence of the premises.

All three methods of reasoning, analogy, induction, and deduction, and other methods we could describe, are commonly employed. There is one essential difference, however, between deduction on the one hand and all other methods of reasoning on the other. Whereas the conclusion drawn by analogy or induction has only a probability of being correct, the conclusion drawn by deduction necessarily holds. Thus one might argue that because lions are similar to cows and cows eat grass, lions also eat grass. This argument by analogy leads to a false conclusion. The same is true for induction: although experiment may indeed show that two dozen different substances expand when heated, it does not necessarily follow that all substances do. Thus water, for example, when heated from 0° to 4° centigrade[\*](#pg41fn1) does not expand; it contracts.

Since deductive reasoning has the outstanding advantage of yielding an indubitable conclusion, it would seem obvious that one should always use this method in preference to the others. But the situation is not that simple. For one thing analogy and induction are often easier to employ. In the case of analogy, a similar situation may be readily available. In the case of induction, experience often supplies the facts with no effort at all. The fact that the sun rises every morning is noticed by all of us almost automatically. Furthermore, deductive reasoning calls for premises which it may be impossible to obtain despite all efforts. Fortunately we can use deductive reasoning in a variety of situations. For example, we can use it to find the distance to the moon. In this instance, both analogy and induction are powerless, whereas, as we shall see later, deduction will obtain the result quickly. It is also apparent that where deduction can replace induction based on expensive experimentation, deduction is preferred.

Because we shall be concerned primarily with deductive reasoning, let us become a little more familiar with it. We have given several examples of deductive reasoning and have asserted that the conclusions are inescapable consequences of the premises. Let us consider, however, the following example. We shall accept as premises that

All good cars are expensive

and

All Locomobiles are expensive.

We might conclude that

All Locomobiles are good cars.

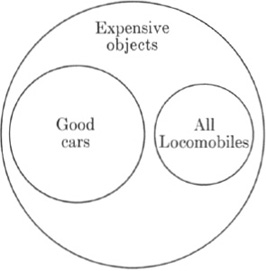


Fig. 3–4.



Fig. 3–5.

The reasoning here is intended as deductive; that is, the presumption in drawing this conclusion is that it is an inevitable consequence of the premises. Unfortunately, the reasoning is not correct. How can we see that it is not correct? A good way of picturing deductive arguments which enables us to see whether or not they are correct is called the circle test.

We note that the first premise deals with cars and expensive objects. Let us think of all the expensive objects in this world as represented by the points of a circle, the largest circle in [Fig. 3–4](#fig3_4). The statement that all good cars are expensive means that all good cars are a part of the collection of expensive objects. Hence we draw another circle within the circle of expensive objects, and the points of this smaller circle represent all the good cars. The second premise says that all Locomobiles are expensive. Hence if we represent all Locomobiles by the points of a circle, this circle, too, must be drawn within the circle of expensive objects. However we do not know, on the basis of the two premises, where to place the circle representing all Locomobiles. It can, as far as we know, fall in the position shown in the figure. Then we cannot conclude that all Locomobiles are good cars, because if that conclusion were inevitable, the circle representing Locomobiles must fall inside the circle representing good cars.

Many people do conclude from the above premises that all Locomobiles are good cars and the reason that they err is that they confuse the premise “All good cars are expensive” with the statement that “All expensive cars are good.” Were the latter statement our first premise then the deductive argument would be valid or correct.

Let us consider another example. Suppose we take as our premises that

All professors are learned people

and

Some professors are intelligent people.

May we necessarily conclude that

Some intelligent people are learned?

It may or may not be obvious that this conclusion is correct. Let us use the circle test. We draw a circle representing the class of learned people ([Fig 3–5](#fig3_5)). Since the first premise tells us that all professors are learned people, the circle representing the class of professors must fall within the circle representing learned people. The second premise introduces the class of intelligent people, and we now have to determine where to draw that circle. This class must include some professors. Hence the circle must intersect the circle of professors. Since the latter is inside the circle of learned people, some intelligent people must fall within the class of learned people.

These examples of deductive reasoning may make another point clear. In determining whether a given argument is correct or valid, we must rely only upon the facts given in the premises. We may not use information which is not explicitly there. For example, we may believe that learned people are intelligent because to acquire learning they must possess intelligence. But this belief or fact, if it is a fact, cannot enter into the argument. Nothing that one may happen to know or believe about learned or intelligent people is to be used unless explicitly stated in the premises. In fact, as far as the validity of the argument is concerned, we might just as well have considered the premises

All x’s are y’s,

Some x’s are z’s,

and the conclusion, then, is

Some z’s are y’s.

Here we have used x for professor, y for learned person, and z for intelligent person. The use of x, y, and z does make the argument more abstract and more difficult to retain in the mind, but it emphasizes that we must look only at the information in the premises and avoids bringing in extraneous information about professors, learned people, and intelligent people. When we write the argument in this more abstract form, we also see more clearly that what determines the validity of the argument is the form of the premises rather than the meaning of x, y, and z.

A great deal of deductive reasoning falls into the patterns we have been illustrating. There are, however, variations that should be noted. It is quite customary, especially in the geometry we learn in high school, to state theorems in what is called the “if . . . then” form. Thus one might say, if a triangle is isosceles, then its base angles are equal. One could as well say, all isosceles triangles have equal base angles; or, the base angles of an isosceles triangle are equal. All three versions say the same thing.

Connected with the “if . . . then” form of a premise is a related statement which is often misunderstood. The statement “if a man is a professor, he is learned” offers no difficulty. As noted in the preceding paragraph, it is equivalent to “all professors are learned.” However the statement “only if a man is a professor, is he learned” has quite a different meaning. It means that to be learned one must be a professor or that if a man is learned, he must be a professor. Thus the addition of the word only has the significance of interchanging the “if” clause and the “then” clause.

We shall encounter numerous instances of deductive reasoning in our work. The subject of deductive reasoning is customarily studied in logic, a discipline which treats more thoroughly the valid forms of reasoning. However, we shall not need to depend upon formal training in logic. In most cases, common experience will enable us to ascertain whether the reasoning is or is not valid. When in doubt, we can use the circle test. Moreover, mathematics itself is the superb field from which to learn reasoning and is the best exercise in logic. The laws of logic were in fact formulated by the Greeks on the basis of their experiences with mathematical arguments.

#### EXERCISES

1. A coin is tossed ten times and each time it falls heads. What conclusion does inductive reasoning warrant?

2. Characterize deductive reasoning.

3. What superior features does deductive reasoning possess compared with induction and analogy?

4. Can you prove deductively that George Washington was the best president of the United States?

5. Can one always apply deductive reasoning to prove a desired statement?

6. Can you prove deductively that monogamy is the best system of marriage?

7. Are the following purportedly deductive arguments valid?

a) All good cars are expensive. A Daffy is an expensive car. Therefore a Daffy is a good car.

b) All New Yorkers are good citizens. All good citizens give to charity. Therefore all New Yorkers give to charity.

c) All college students are clever. All young boys are clever. Therefore all young boys are college students.

d) The same premises as in (c), but the conclusion: All college students are young boys.

e) It rains every Monday and it is raining today; hence today must be Monday.

f) No decent people curse; Americans are decent; therefore Americans do not curse.

g) No decent people curse; Americans curse; therefore some Americans are not decent.

h) No decent people curse; some Americans are not decent; therefore some Americans curse.

i) No undergraduates have a bachelor-of-arts degree; no freshmen have a bachelor-of-arts degree. Therefore all freshmen are undergraduates.

8. If someone gave you a valid deductive argument but the conclusion was not true, where would you look for the difficulty?

9. Distinguish between the validity of a deductive argument and the truth of the conclusion.

3–5 MATHEMATICAL PROOF

We have seen so far in our discussion of reasoning that there are several methods of reasoning and that all are useful. These methods can be applied to mathematical problems. Let us suppose that one wished to determine the sum of the angles of a triangle. He could draw on paper many different triangles or construct some out of wood or metal and measure the angles. In each case he would find that the sum is as close to 180° as the eye and hand can determine. By inductive reasoning he could conclude that the sum of the angles in every triangle is 180°. As a matter of fact, the Babylonians and Egyptians did in effect use inductive reasoning to establish their mathematical results. They must have determined by measurement that the area of a triangle is one-half the base times the altitude and, having used this formula repeatedly and having obtained reliable results, they concluded that the formula is correct.

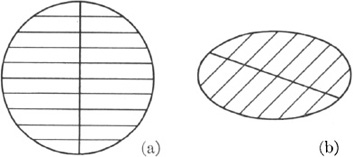


Fig. 3–6.  
The mid-points of parallel chords lie on a straight line.

To see that reasoning by analogy can be used in mathematics, let us note first that the centers of a set of parallel chords of a circle lie on a straight line ([Fig. 3–6a](#fig3_6)). In fact this line is a diameter of the circle. Now an ellipse ([Fig. 3–6b](#fig3_6)) is very much like a circle. Hence one might conclude that the centers of a set of parallel chords of an ellipse also lie on a straight line.

Deduction is certainly applicable in mathematics. The proofs which one learns in Euclidean geometry are deductive. As another illustration we might consider the following algebraic argument. Suppose one wishes to solve the equation x − 3 = 7. One knows that equals added to equals give equals. If we added 3 to both sides of the preceding equation, we would be adding equals to an equality. Hence the addition of 3 to both sides is justified. When this is done, the result is x = 10, and the equation is solved.

Thus all three methods are applicable. There is a lot to be said for the use of induction and analogy. The inductive argument for the sum of the angles of a triangle can be carried out in a matter of minutes. The argument by analogy given above is also readily made. On the other hand, finding deductive proofs for these same conclusions might take weeks or might never be accomplished by the average person. As a matter of fact, we shall soon encounter some examples of conjectures for which the inductive evidence is overwhelming but for which no deductive proof has been thus far obtained even by the best mathematicians.

Despite the usefulness and advantages of induction and analogy, mathematics does not rely upon these methods to establish its conclusions. All mathematical proofs must be deductive. Each proof is a chain of deductive arguments, each of which has its premises and conclusion.

Before examining the reasons for this restriction to deductive proof, we might contrast the method of mathematics with those of the physical and social sciences. The scientist feels free to draw conclusions by any method of reasoning and, for that matter, on the basis of observation, experimentation, and experience. He may reason by analogy as, for example, when he reasons about sound waves by observing water waves or when he reasons about a possible cure for a disease affecting human beings by testing the cure on animals. In fact reasoning by analogy is a powerful method in science. The scientist may also reason inductively: if he observes many times that hydrogen and oxygen combine to form water, he will conclude that this combination will always form water. At some stages of his work the scientist may also reason deductively and, in fact, even employ the concepts and methods of mathematics proper.

To contrast further the method of mathematics with that of the scientist—and perhaps to illustrate just how stubborn the mathematician can be—we might consider a rather famous example. Mathematicians are concerned with whole numbers, or integers, and among these they distinguish the prime numbers. A prime is a number which has no integral divisors other than itself and 1. Thus 11 is a prime number, whereas 12 is not because it is divisible by 2 for example. Now by actual trial one finds that each of the first few even numbers can be expressed as the sum of two prime numbers. For example, 2 = 1 + 1; 4 = 2 + 2; 6 = 3 + 3; 8 = 3 + 5; 10 = 3 + 7; . . . . If one investigates larger and larger even numbers, one finds without exception that every even number can be expressed as the sum of two primes. Hence by inductive reasoning one could conclude that every even number is the sum of two prime numbers.

But the mathematician does not accept this conclusion as a theorem of mathematics because it has not been proved deductively from acceptable premises. The conjecture that every even number is the sum of two primes, known as Goldbach’s hypothesis because it was first suggested by the eighteenth-century mathematician Christian Goldbach, is an unsolved problem of mathematics. The mathematician will insist on a deductive proof even if it takes thousands of years, as it literally has in some instances, to find one. However a scientist would not hesitate to use this inductively well supported conclusion.

Of course, the scientist should not be surprised to find that some of his conclusions are false because, as we have seen, induction and analogy do not lead to sure conclusions. But it does seem as though the scientist’s procedure is wiser since he can take advantage of any method of reasoning which will help him advance his knowledge. The mathematician by comparison appears to be narrow-minded or shortsighted. He achieves a reputation for certainty, but at the price of limiting his results to those which can be established deductively. How wise the mathematician may be in his insistence on deductive proof we shall learn as we proceed.

The decision to confine mathematical proof to deductive reasoning was made by the Greeks of the classical period. And they not only rejected all other methods of proof in mathematics, but they also discarded all the knowledge which the Egyptians and Babylonians had acquired over a period of four thousand years because it had only an empirical justification. Why did the Greeks do it?

The intellectuals of the classical Greek period were largely absorbed in philosophy and these same men, because they possessed intellectual interests, were the very ones who developed mathematics as a system of thought. The Ionians, the Pythagoreans, the Sophists, the Platonists, and the Aristotelians were the leading philosophers who gave mathematics its definitive form. The credit for initiating this step probably belongs to one school of Greek philosopher-mathematicians, known as the Ionian school. However, if credit can be assigned to any one person, it belongs to Thales, who lived about 600 B.C. Though a native of Miletus, a Greek city in Asia Minor, Thales spent many years in Egypt as a merchant. There he learned what the Egyptians had to offer in mathematics and science, but apparently he was not satisfied, for he would accept no results that could not be established by deductive reasoning from clearly acceptable axioms. In his wisdom Thales perceived what we shall perceive as we follow the story of mathematics, that the obvious is far more suspect than the abstruse.

Thales probably supplied the proof of many geometrical theorems. He acquired great fame as an astronomer and is believed to have predicted an eclipse of the sun in 585 B.C. A philosopher-astronomer-mathematician might readily be accused of being an impractical stargazer, but Aristotle tells us otherwise. In a year when olives promised to be plentiful, Thales shrewdly cornered all the oil presses to be found in Miletus and in Chios. When the olives were ripe for pressing, Thales was in a position to rent out the presses at his own price. Thales might perhaps have lived in history as a leading businessman, but he is far better known as the father of Greek philosophy and mathematics. From his time onward, deductive proof became the standard in mathematics.

It is to be expected that philosophers would favor deductive reasoning. Whereas scientists select particular phenomena for observation and experimentation and then draw conclusions by induction or analogy, philosophers are concerned with broad knowledge about man and the physical world. To establish universal truths, such as that man is basically good, that the world is designed, or that man’s life has purpose, deductive reasoning from acceptable principles is far more feasible than induction or analogy. As Plato put it in his Republic, “If persons cannot give or receive a reason, they cannot attain that knowledge which, as we have said, man ought to have.”

There is another reason that philosophers favor deductive reasoning. These men seek truths, the eternal verities. We have seen that of all the methods of reasoning only deductive reasoning grants sure and exact conclusions. Hence this is the method which philosophers would almost necessarily adopt. Not only do induction and analogy fail to yield absolutely unquestionable conclusions, but many Greek philosophers would not have accepted as facts the data with which these methods operate, because these are acquired by the senses. Plato stressed the unreliability of sensory perceptions. Empirical knowledge, as Plato put it, yields opinion only.

The Greek preference for deduction had a sociological basis. Contrary to our own society wherein bankers and industrialists are respected most, in classical Greek society, the philosophers, mathematicians, and artists were the leading citizens. The upper class regarded earning a living as an unfortunate necessity. Work robbed man of time and energy for intellectual activities, the duties of citizenship and discussion. These Greeks did not hesitate to express their disdain for work and business. The Pythagoreans, who, as we shall see, delighted in the properties of numbers and applied numbers to the study of nature, derided the use of numbers in commerce. They boasted that they sought knowledge rather than wealth. Plato, too, maintained that knowledge rather than trade was the goal in studying arithmetic. Freemen, he declared, who allowed themselves to become preoccupied with business should be punished, and a civilization which is concerned mainly with the material wants of man is no more than a “city of happy pigs.” Xenophon, the famous Greek general and historian, says, “What are called the mechanical arts carry a social stigma and are rightly dishonored in our cities.” Aristotle wanted an ideal society in which citizens would not have to practice any mechanical arts. Among the Boeotians, one of the independent tribes of Ancient Greece, those who defiled themselves, with commerce were by law excluded from state positions for ten years.

Who did the daily work of providing food, shelter, clothing, and the other necessities of life? Slaves and free men ineligible for citizenship ran the businesses and the households, did unskilled and technical work, managed the industries, and carried on the professions such as medicine. They produced even the articles of refinement and luxury.

In view of this attitude of the Greek upper class towards commerce and trade, it is not hard to understand the classical Greek’s preference for deduction. People who do not “live” in the workaday world can learn little from experience, and people who will not observe and use their hands to experiment will not have the facts on which to base reasoning by analogy or induction. In fact the institution of slavery in classical Greek society fostered a divorce of theory from practice and favored the development of speculative and deductive science and mathematics at the expense of experimentation and practical applications.

Over and above the various cultural forces which inclined the Greeks toward deduction were a farsightedness and a wisdom which mark true genius. The Greeks were the first to recognize the power of reason. The mind was a faculty not only additional to the senses but more powerful than the senses. The mind can survey all the whole numbers, but the senses are limited to perceiving only a few at a time. The mind can encompass the earth and the heavens; the sense of sight is confined to a small angle of vision. Indeed the mind can predict future events which the senses of contemporaries will not live to perceive. This mental faculty could be exploited. The Greeks saw clearly that if man could obtain some truths, he could establish others entirely by reasoning, and these new truths, together with the original ones, enabled man to establish still other truths. Indeed the possibilities would multiply at an enormous rate. Here was a means of acquiring knowledge which had been either overlooked or neglected.

This was indeed the plan which the Greeks projected for mathematics. By starting with some truths about numbers and geometrical figures they could deduce others. A chain of deductions might lead to a significant new fact which would be labeled a theorem to call attention to its importance. Each theorem added to the stock of truths that could serve as premises for new deductive arguments, and so one could build an immense body of knowledge about the basic concepts.

Although the Greeks may have been guilty of overemphasizing the power of the mind unaided by experience and observation to obtain truths, there is no doubt that in insisting on deductive proof as the sole method, they rose above the practical level of carpenters, surveyors, farmers, and navigators. At the same time they elevated the subject of mathematics to a system of thought. Moreover the preference for reason which they exhibited gave this faculty the high prestige which it now enjoys and permitted it to exercise its true powers. When we have surveyed some of the creations of the mind that succeeding civilizations building on the Greek plan contributed, we shall appreciate the true depth of the Greek vision.

#### EXERCISES

1. Compare Greek and pre-Greek standards of proof in mathematics. Reread the relevant parts of [Chapter 2](#ch2).

2. Distinguish science and mathematics with respect to ways of establishing conclusions.

3. Explain the statement that the Greeks converted mathematics from an empirical science to a deductive system.

4. Are the following deductive arguments valid?

a) All even numbers are divisible by 4. Ten is an even number. Hence 10 is divisible by 4.

b) Equals divided by equals give equals. Dividing both sides of 3x = 6 by 3 is dividing equals by equals. Hence x = 2.

5. Does it follow from the fact that the square of any odd number is odd that the square of any even number is even?

6. Criticize the argument:

The square of every even number is even because 22 = 4, 42 = 16, 62 = 36, and it is obvious that the square of any larger even number also is even.

7. If we accept the premises that the square of any odd number is odd and the square of any even number is even, does it follow deductively that if the square of a number is even, the number must be even?

8. Why did the Greeks insist on deductive proof in mathematics?

9. Let us take for granted that if a triangle has two equal sides, the opposite angles are equal and that we have a triangle in which all three sides are equal. Prove deductively that all three angles are equal in the triangle under consideration. You may also use the premise that things equal to the same thing are equal to one another.

10. How did the Greeks propose to obtain new truths from known ones?

3–6 AXIOMS AND DEFINITIONS

From our discussion of deductive reasoning we know that to apply such reasoning we must have premises. Hence the question arises, what premises does the mathematician use? Since the mathematician reasons about numbers and geometrical figures, he must of course have facts about these concepts. These cannot be obtained deductively because then there would have to be prior premises, and if one continued this process backward, there would be no starting point. The Greeks readily found premises. It seemed indisputable, for example, that two points determine one and only one line and that equals added to equals give equals.

To the Greeks the premises on which mathematics was to be built were self-evident truths, and they called these premises axioms. Socrates and Plato believed, as did many later philosophers, that these truths were already in our minds at birth and that we had but to recall them. And since the Greeks believed that axioms were truths and since deductive reasoning yielded unquestionable conclusions, they also believed that theorems were truths. This view is no longer held, and we shall see later in this book why mathematicians abandoned it. We now know that axioms are suggested by experience and observation. Naturally, to be as certain as we can of these axioms we select those facts which seem clearest and most reliable in our experience. But we must recognize that there is no guarantee that we have selected truths about the world. Some mathematicians prefer to use the word assumptions instead of axioms to emphasize this point.

The mathematician also takes care to state his axioms at the outset and to be sure as he performs his reasoning that no assumptions or facts are used which were not so stated. There is an interesting story told by former President Charles W. Eliot of Harvard which illustrates the likelihood of introducing unwarranted premises. He entered a crowded restaurant and handed his hat to the doorman. When he came out, the doorman at once picked Eliot’s hat out of hundreds on the racks and gave it to him. He was amazed that the doorman could remember so well and asked him, “How did you know that was my hat?” “I didn’t,” replied the doorman. “Why, then, did you hand it to me?” The doorman’s reply was, “Because you handed it to me, sir.”

Undoubtedly no harm would have been done if the doorman had assumed that the hat he returned to President Eliot belonged to the man. But the mathematician interested in obtaining conclusions about the physical world might be wasting his time if he unwittingly introduced an assumption that he had no right to make

There is one other element in the logical structure of mathematics about which we shall say a few words now and return to in a later chapter ([Chapter 20](#ch20)). Like other studies mathematics uses definitions. Whenever we have occasion to use a concept whose description requires a lengthy statement, we introduce a single word or phrase to replace that lengthy statement. For example, we may wish to talk about the figure which consists of three distinct points which do not lie on the same straight line and of the line segments joining these points. It is convenient to introduce the word triangle to represent this long description. Likewise the word circle represents the set of all points which are at a fixed distance from a definite point. The definite point is called the center, and the fixed distance is called the radius. Definitions promote brevity.

#### EXERCISES

1. What belief did the Greeks hold about the axioms of mathematics?

2. Summarize the changes which the Greeks made in the nature of mathematics.

3. Is it fair to say that mathematics is the child of philosophy?

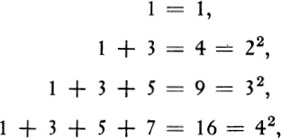
3–7 THE CREATION OF MATHEMATICS

Because mathematical proof is strictly deductive and merely reasonable or appealing arguments may not be used to establish a conclusion, mathematics has been described as a deductive science, or as the science which derives necessary conclusions, that is, conclusions which necessarily or inevitably follow from the axioms. This description of mathematics is incomplete. Mathematicians must also discover what to prove and how to go about establishing proofs. These processes are also part of mathematics and they are not deductive.

How does the mathematician discover what to prove and the deductive arguments that lead to the conclusions? The most fertile source of mathematical ideas is nature herself. Mathematics is devoted to the study of the physical world, and simple experience or the more careful scrutiny of nature suggests idea after idea. Let us consider here a few simple examples. Once mathematicians had decided to devote themselves to geometric forms, it was only natural that such questions should arise as, what are the area, perimeter, and sum of the angles of common figures? Moreover, it is even possible to see how the precise statement of the theorem to be proved would follow from direct experience with physical objects. The mathematician might measure the sum of the angles of various triangles and find that these measurements all yield results close to 180°. Hence the suggestion that the sum of the angles in every triangle is 180° occurs as a possible theorem. To decide the question, which has more area, a polygon or a circle having the same perimeter, one might cut out cardboard figures and weigh them. The relative weights would suggest the statement of the theorem to be proved.

After some theorems have been suggested by direct physical problems, others are readily conceived by generalizing or varying the conditions. Thus knowing the problem of determining the sum of the angles of a triangle, one might ask, What is the sum of the angles of a quadrilateral, a pentagon, and so forth? That is, once the mathematician begins an investigation which is suggested by a physical problem, he can easily find new problems which go beyond the original one.

In the domains of arithmetic and algebra direct calculation with numbers, which is analogous to measurement in geometry, will suggest possible theorems. Anyone who has played with integers, for example, has doubtless observed the following facts:



We note that each number on the right is the square of the number of odd numbers appearing on the left; thus in the fourth line, there are four numbers on the left side, and the right side is 42. The general result which these calculations suggest is that if the first n odd numbers were on the left side, then the sum would be n2. Of course, this possible theorem is not proved by the above calculations. Nor could it ever be proved by such calculations, for no mortal man could make the infinite set of computations required to establish the conclusion for every n. The calculations do, however, give the mathematician something to work on.

These simple illustrations of how observation, measurement, and calculation suggest possible theorems are not too striking or very profound. We shall see in the course of later work how physical problems suggest more significant mathematical theorems. However, experience, measurement, calculation, and generalization do not include the most fertile source of possible theorems. And it is especially true in seeking methods of proof that more than routine techniques must be utilized. In both endeavors the most important source is the creative act of the human mind.

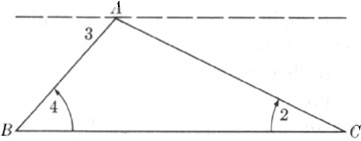


Fig. 3–7

Let us consider the matter of proof. Suppose one has discovered by measurements that the sum of the angles of various triangles is 180°. One must now prove this result deductively. No obvious method will do the job. Some new idea is required, and the reader who remembers his elementary geometry will recall that the proof is usually made by drawing a line through one vertex (A in [Fig. 3–7](#fig3_7)) and parallel to the opposite side. It then turns out as a consequence of a previously established theorem on parallel lines that the angles 1 and 2 are equal, as are the angles 3 and 4. However the angles 1, 3, and the angle A of the triangle itself do add up to 180°, and so the same is true for the angles of the triangle. This method of proof is not routine. The idea of drawing the line through A must be supplied by the mind. Some methods of proof seem so devious and artificial that they have provoked critical comments. The philosopher Arthur Schopenhauer called Euclid’s proof of the Pythagorean theorem “a mouse-trap proof” and “a proof walking on stilts, nay, a mean, underhand proof.”

The above example has been offered to emphasize the fact that ingenious mathematical work must be done in finding methods of proofs even after the question of what to prove is disposed of. In the search for a method of proof, as in finding what to prove, the mathematician must use audacious imagination, insight, and creative ability. His mind must see possible lines of attack where others would not. In the domains of algebra, calculus, and advanced analysis especially, the first-rate mathematician depends upon the kind of inspiration that we usually associate with the creation of music, literature, or art. The composer feels that he has a theme which when properly developed will produce true music. Experience and a knowledge of music aid him in arriving at this conviction. Similarly, the mathematician surmises that he has a conclusion which will follow from the axioms of mathematics. Experience and knowledge may guide his thoughts into the proper channels. Modifications of one sort or another may be required before a correct proof and a satisfactory statement of the new theorem are achieved. But essentially both mathematician and composer are moved by an afflatus which enables them to see the final edifice before a single stone is laid.

We do not know just what mental processes may lead to correct insight any more than we know how it was possible for Keats to write fine poetry or why Rembrandt was able to turn out fine paintings. One might say of mathematical creation what P. W. Bridgman, the noted physicist, has said of scientific method, that it consists of “doing one’s damnedest with one’s mind, no holds barred.” There is no logic or infallible guide which tells the mind how to think. The very fact that many great mathematicians have tackled a problem and failed and that another comes along and solves it shows that the mind has something to contribute.

The preceding discussion of the creation of mathematics should correct several mistaken popular impressions. When creating a mathematical proof, the mind does not see the cold, ordered arguments which one reads in texts, but rather it perceives an idea or a scheme which when properly formulated constitutes the deductive proof. The formal proof, so to speak, merely sanctions the conquest already made by the intuition. Secondly, the deductive proof is not the preferable form by which to grasp the idea or method employed. In fact the deductive argument often conceals the idea because the logical form is not perspicuous to the intuition. At the very least the details of the arguments obscure the main threads. The value of the deductive organization of the proof is that it enables the creator and the reader to test the arguments by the standards of exact reasoning. Thirdly, there is the prevalent but mistaken notion that scientists and mathematicians must keep their minds open and unbiased in pursuing an investigation. They are not supposed to prejudge the conclusion. Actually the mathematician must first decide what to prove, and this conclusion not only does but must precede the search for the proof, or else he would not know where to head. This is not to say that the mathematician may not sometimes make a false conjecture. If he does, his search for a proof will fail or in the course of the search he will realize that he cannot prove what he is after, and he will correct his conjecture. But in any case he knows what he is trying to prove.

#### EXERCISES

1. Consider the parallelogram ABCD ([Fig. 3–8](#fig3_8)). By definition, the opposite sides are parallel. Now introduce the diagonal BD. Does observation suggest a possible theorem relating the triangles ABD and BDC?

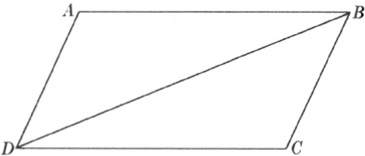


Fig. 3–8

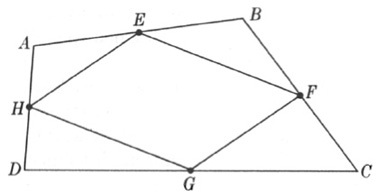


Fig. 3–9

2. Consider any quadrilateral ABCD ([Fig. 3–9](#fig3_9)) and the figure formed by joining the mid-points E, F, G, H of the sides of the quadrilateral. Does observation or intuition suggest any significant fact about the quadrilateral EFGH?

3. The formula n2 − n + 41 is supposed to yield primes for various values of n. Thus when n = 1,

12 − 1 + 41 = 41,

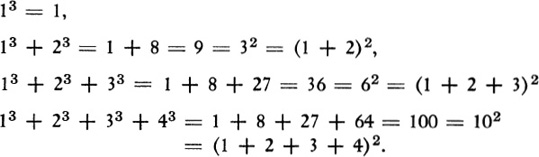
and this is a prime. When n = 2,

22 − 2 + 41 = 43,

and this is a prime. Test the formula for n = 3 and n = 4. Are the resulting values of the formula primes? Have you proved, then, that the formula always yields primes?

4. Can you specify conditions under which two quadrilaterals will be congruent, that is, have the same size and shape?

5. The following lines show some calculations with the sum of the cubes of whole numbers:



What generalization do these few calculations suggest?

#### REVIEW EXERCISES

1. What basis did the Egyptians and Babylonians have for believing in the correctness of their mathematical conclusions?

2. Compare Greek and pre-Greek understanding of the concepts of mathematics.

3. What was the Greek plan for establishing mathematical conclusions?

4. In what sense is mathematics a creation of the Greeks rather than of the Egyptians and the Babylonians?

5. Suppose we accept the premises that all professors are intelligent people and all professors are learned people. Which of the following conclusions is validly deduced?

a) Some intelligent people are learned.

b) Some learned people are intelligent.

c) All intelligent people are learned.

d) All learned people are intelligent.

6. Suppose we accept the premises that all college students are wise, and no professors are college students. Which of the following conclusions is validly deduced?

a) No professors are wise.

b) Some professors are wise.

c) All professors are wise.

7. Is the following argument valid?

All parallelograms are quadrilaterals, and figure ABCD is a quadrilateral. Hence figure ABCD is a parallelogram.

8. What conclusion can you deduce from the premises,

Every successful student must work hard,

and

John does not work hard?

9. Smith says,

If it rains I go to the movies.

If Smith went to the movies, what can you conclude deductively?

10. Smith says,

I go to the movies only if it rains.

If Smith went to the movies, what can you conclude deductively?

##### Topics for Further Investigation

To pursue any of these topics use the books listed below under Recommended Reading.

1. The life and work of the Pythagoreans

2. The life and work of Euclid

##### Recommended Reading

BELL, ERIC T.: The Development of Mathematics, 2nd ed., Chaps. 2 and 3, McGraw-Hill Book Co., N.Y., 1945.

BELL, ERIC T.: Men of Mathematics, Simon and Schuster, New York, 1937.

CLAGETT, MARSHALL: Greek Science in Antiquity, Chap. 2, Abelard-Schuman, Inc., New York, 1955.

COHEN, MORRIS R. and E. NAGEL: An Introduction to Logic and Scientific Method, Chaps. 1 through 5, Harcourt Brace and Co., New York, 1934.

COOLIDGE, J. L.: The Mathematics of Great Amateurs, Chap. 1, Dover Publications, Inc., New York, 1963.

HAMILTON, EDITH: The Greek Way to Western Civilization, Chaps. 1 through 3, The New American Library, New York, 1948.

JEANS, SIR JAMES: The Growth of Physical Science, 2nd ed., Chap. 2, Cambridge University Press, Cambridge, 1951.

NEUGEBAUER, OTTO: The Exact Sciences in Antiquity, Princeton University Press, Princeton, 1952.

SMITH, DAVID EUGENE: History of Mathematics, Vol. I., Dover Publications, Inc., New York, 1958.

STRUIK, DIRK J.: A Concise History of Mathematics, Dover Publications, Inc., New York, 1948.

TAYLOR, HENRY OSBORN: Ancient Ideals, 2nd ed., Vol. I, Chaps. 7 through 13, The Macmillan Co., New York, 1913.

WEDBERG, ANDERS: Plato’s Philosophy of Mathematics, Almqvist and Wiksell, Stockholm, 1955 (for students of philosophy).

[\*](#ipg41fn1) In scientific texts, “celsius” is considered to be the more precise term.

## CHAPTER 4

NUMBER: THE FUNDAMENTAL CONCEPT

A marvelous neutrality have these things Mathematical, and also a strange participation between things supernatural, immortal, intellectual, simple, and indivisible, and things natural, mortal, sensible, compounded and divisible.

JOHN DEE (1527–1608)

4–1 INTRODUCTION

Just as we are inclined to accept the sun, moon, and stars as our birthright and do not appreciate the grandeur, the mystery, and the knowledge which can be gleaned from the contemplation of the heavens, so are we inclined to accept our number system. There is, however, this difference. Many of us would not claim the latter and would gladly sell it for a mess of pottage. Because we are forced to learn about numbers and operations with numbers while we are still too young to appreciate them—a preparation for life which hardly excites our interest in the future—we grow up believing that numbers are drab and uninteresting. But the number system warrants attention not only as the basis of mathematics, but because it contains weighty and beautiful ideas which lend themselves to powerful applications.

Among past civilizations, the Greeks best appreciated the wonder and power of the concept of number. They were, of course, a people with great intellectual perception, but perhaps because they viewed numbers abstractly, they saw more clearly their true nature. The very fact that one can abstract from many diverse collections of objects a property such as “fiveness” struck the Greeks as a marvelous discovery. If one may use the ridiculous to accentuate the sublime, one may say that the Greek delight in numbers was the rational counterpart of the hysteria which many young and old Americans experience when they encounter numbers in the form of baseball scores and batting averages.

4–2 WHOLE NUMBERS AND FRACTIONS

The first Greeks who, to our knowledge, expressed their satisfaction with numbers and propounded a philosophy based on numbers which is extremely alive and vital today were the Pythagoreans. This group was founded in the middle of the sixth century B.C. by Pythagoras. We know rather little that is certain about this man. However, it seems very likely that he was born in 569 B.C. in a Greek settlement on the island of Samos in the Aegean Sea. Like many other Greeks he traveled to Egypt and to the Near East to learn what these older civilizations had to offer, and then settled in Croton, another Greek city in southern Italy. Pythagoras and his followers were among the early founders of the great Greek civilization, and so it is not surprising to find that the rational attitude which characterizes the Greeks was still surrounded in his times with mystical and religious doctrines prevalent in Egypt and its eastern neighbors. In fact the Pythagoreans were a religious sect as well as students of philosophy and nature.

Membership in the group was restricted, and the members were pledged to secrecy. Among their religious doctrines was the belief that the soul was tainted by the body. To purify the soul they maintained celibacy; their religious practices were also supposed to be efficacious in purifying the soul. At death the soul was reincarnated in another human or an animal. Like most mystics they observed certain taboos. They would not touch a white cock, walk on the highways, use iron to stir a fire, or leave the marks of ashes on a pot.

The secrecy of the group, its aloofness, and an attempted interference in the political affairs of Croton finally aroused the people of this city to drive out the Pythagoreans. We do not know for certain what happened to Pythagoras. One story has it that he fled to Metapontum, another Greek city in southern Italy, and was murdered there. However, the Pythagoreans continued to be influential in Greek intellectual life. One of their notable members was the philosopher Plato.

The Pythagoreans were impressed with numbers and, because they were mystics, attached to the whole numbers meanings and significances which we now regard as childish. Thus, they considered the number “one” as the essence or very nature of reason, for reason could produce only one consistent body of doctrines. The number “two” was identified with opinion, clearly because the very meaning of opinion implies the possibility of an opposing opinion, and thus of at least two. “Four” was identified with justice because it is the first number which is the product of equals. Of course, one can also be thought of as 1 times 1, but to the Pythagoreans one was not a number in the full sense because it did not represent quantity. The Pythagoreans represented numbers as dots in sand or by pebbles, and for each number the dots or pebbles had a special arrangement. Thus the number “four” was pictured as four dots suggesting a square, and so the square and justice were also linked. Foursquare and square shooter still mean a person who acts justly. “Five” signified marriage because it was the union of the first masculine number, three, and the first feminine number, two. (Odd numbers were masculine and even numbers feminine.) The number “seven” represented health and “eight,” friendship or love.

We shall not pursue all the ideas which the Pythagoreans developed about numbers. What is significant about their work is that they were the first to study properties of whole numbers. As we shall see in a later chapter, they also possessed the vision of deep mystics and saw that numbers could be used to represent and even embody the essence of natural phenomena.

The speculations and results obtained by the Pythagoreans about whole numbers and ratios of whole numbers, or fractions as we prefer to call them, were the beginning of a long and involved development of arithmetic as a science as opposed to arithmetic as a tool for daily applications. During the 2500 years since the Pythagoreans first called attention to the importance of numbers, man has not only learned to better appreciate the idea but has invented excellent methods of writing quantity and of performing the four operations of arithmetic, i.e., ambition, distraction, uglification, and derision, as Lewis Carroll called them. While these methods of writing and operating with numbers are largely familiar, there are a few facts which are worthy of comment.

One of the most important members of our present number system is the mathematical representation of no quantity, that is, zero. We are accustomed to this number and yet usually fail to appreciate two facts about it. The first is that this member of our number system came rather late. The idea of using zero was conceived by the Hindus and, like other of their ideas, reached Europe through the Arabs. It had not occurred to earlier civilizations, even to the Greeks, that it would be useful to have a number which represents the absence of any objects. Connected with this late appearance of the number is the second significant fact, namely, that zero must be distinguished from nothing. Undoubtedly it was the inability of earlier peoples to perceive this distinction which accounts for their failure to introduce the zero. That zero must be distinguished from nothing is easily seen from several examples. A student’s grade in a course he never took is no grade or nothing. He may, however, have the grade of zero in a course he has taken. If a person has no account in a bank, his balance is nothing. If he has a bank account, he may very well have a balance of zero.

Because zero is a number, we may operate with it; for example, we may add zero to another number. Thus 5 + 0 = 5. By contrast 5 + nothing is meaningless or nothing. The only restriction on zero as a number is that one cannot divide by zero. Division by zero does, so to speak, produce nothing. Because so many false steps in mathematics result from division by zero, it is well to understand clearly why we cannot do this. The answer to a problem of division, say image, is some number which when multiplied by the divisor yields the dividend. In our example, 3 is the answer because 3 · 2 = 6. Hence the answer to image should be a number which when multiplied by 0 gives 5. However, any number multiplied by 0 gives 0 and not 5. Thus, there is no answer to the problem of image. In the case of image the answer should be some number which when multiplied by 0 yields the dividend 0. However, any number may then serve as a quotient because any number multiplied by 0 gives 0. But mathematics cannot tolerate such an ambiguous situation. If image arises and any number may serve as an answer, we do not know what number to take and hence are not aided. It is as if we asked a person for directions to some place and he replied, Take any direction.

With the availability of zero, mathematicians were finally able to develop our present method of writing whole numbers. First of all we count in units and represent large quantities in tens, tens of tens, tens of tens of tens, etc. Thus we represent two hundred and fifty-two by 252. The left-hand 2 means, of course, two tens of tens; the 5 means 5 times 10; and the right-hand 2 means 2 units. The concept of zero makes such a system of writing quantities practical since it enables us to distinguish 22 and 202. Because ten plays such a fundamental role, our number system is called the decimal system, and ten is called the base. The use of ten resulted most likely from the fact that man counted on his fingers and, when he had used the fingers on his two hands, considered the number arrived at as a larger unit.

Because the position of an integer determines the quantity it represents, the principle involved is called positional notation. The decimal system of positional notation is due to the Hindus; however, the same scheme was used two millenniums earlier by the Babylonians, but with base 60 and in more limited form since they did not have zero.

The operations of arithmetic, addition, subtraction, multiplication, and division, are of course familiar to us, but it is perhaps not recognized that these operations are quite sophisticated and remarkably efficient. They date back to Greek times and gradually evolved, as improvements in the methods of writing numbers and the concept of zero were introduced. The Europeans picked up the methods from the Arabs. Previously the Europeans had used the Roman system of writing numbers, and the operations were based on that system. Partly because these latter methods were relatively cumbersome and partly because education was limited to a few people, those who acquired the art of calculation were regarded as skilled mathematicians. In fact the processes defied the average man so much that it seemed to him that those possessed of the ability must have magical powers. Good calculators were called practitioners of the “Black Art.”

To appreciate the efficiency of our present methods we would have to learn the older ones and even acquire some facility in them, to make the comparison a fair one. But we cannot spare the time and effort. Perhaps the one point we should emphasize is how much our methods of arithmetic depend upon positional notation. This can be seen even in a simple problem of addition. To add 387 and 359 say, the written work is

image

However, in performing this work, we think as follows. We add the units 7 and 9, the “tens” quantities 8 and 5, and the “hundreds” 3 and 3, separately. When we add the 7 and 9, we obtain 16. We recognize that 16 is 1 · 10 + 6, and so we add the 1 · 10 to the 13 · 10 already obtained from the 8 and 5. We say that we “carry” the 1 · 10 over, and instead of 13 · 10 we obtain 14 · 10. However, 14 · 10 is (10 + 4) · 10 or 1 · 102 + 4 · 10. Thus we write 4 in the tens’ column, add the 1 · 102 to the 6 · 102 already obtained from the 3 and 3, and arrive at 7 · 102. All these steps are usually executed rather mechanically by writing the appropriate numbers in the units’, tens’, and hundreds’ places and by using the process called carrying. Were we to analyze the processes of subtraction, multiplication, and division, we would again see how the steps which we learn mechanically in elementary school are just the skeletal processes of thinking suited to positional notation in base ten.

A word about fractions may also be in order. The natural method of writing fractions, for example, image or image, to express parts of a whole presents no difficulties of comprehension. However the operations with fractions do seem to be somewhat arbitrary and mysterious. To add image and image, say, we go through the following process:

image

What we have done is to express each fraction in an equivalent form such that the denominators are now alike, and then add the numerators. We are not required by law to add fractions in this manner. It would, of course, be much simpler if we agreed to add fractions by adding the numerators and adding the denominators so that

image

As a matter of fact, when we multiply two fractions, we do multiply the numerators and multiply the denominators so that it does seem as though the mathematicians prefer to be unnecessarily complicated about the addition of fractions.

The explanation of this seeming mathematical idiosyncrasy is simple: the operations with fractions are formulated to fit experience. When one has image of a pie and image of a pie, he has in all not image but image of a pie. In other words, if mathematical concepts and operations are to fit experience, the nature of the operations is forced upon us. In the case of multiplication of fractions it is correct that multiplying the numerators and multiplying the denominators will yield the fraction which represents the physical result. Thus suppose we had to find image of image, that is image · image. We think of image as 2 · image. Now

image

Then

image

The same result is obtained by multiplying the original numerators and multiplying the original denominators.

The operation of dividing one fraction by another presents a little more difficulty. To see how we arrive at the correct process let us start with some simple examples. Suppose we had to answer the question of how many one-thirds of a pie are in 2 pies. Mathematically this question is formulated as how much is

image

We should note that one bar is larger than the other and the longer bar separates the numerator, the 2, from the denominator image. Now, we know on physical grounds that we can obtain 6 one-thirds from 2 pies. We can obtain this answer arithmetically by inverting the denominator image and multiplying the inverse into the numerator 2. That is,

image

Now let us complicate the problem slightly. How many two-thirds of a pie are contained in 2 pies? Again this question is formulated mathematically as

image

We know on physical grounds that there are 3 two-thirds of a pie in 2 pies. We can obtain this answer arithmetically by inverting the denominator and multiplying this inverse fraction into the numerator. Thus,

image

Now let us complicate the problem still more. We would certainly agree that 2 pies are the same as image pies. If therefore we had to answer the question of how many two-thirds of a pie are contained in image pies, we would know from the preceding example that the answer is 3. How could we obtain this answer directly? The question is, how much is

image

Let us invert the denominator and multiply the inverse into the numerator. Thus

image

Again we see that the process of inverting the denominator and multiplying it into the numerator gives the result which we know on physical grounds is correct.

The significant point, then, is that the rule “to divide one fraction by another, invert the denominator and multiply this inverse into the numerator” is designed to make the mathematical operation give a result which fits experience. This is, of course, the same principle which applies to the other operations. Logically, we may say that we define the operations to be what we have just illustrated for addition, multiplication and division, and in our purely mathematical definitions we do not say anything about agreement with physical facts. But, of course, the definitions would be pointless if they did not give physically correct results.

Fractions, like the whole numbers, can be written in positional notation. Thus

image

If we now agree to suppress the powers of 10, that is 10, 100, and higher powers where they occur, then we can write image. The decimal point reminds us that the first number is really image, the second image, and so forth. The Babylonians had employed positional notation for fractions, but they used base 60 rather than base 10, just as they had for whole numbers. The decimal base for fractions was introduced by sixteenth-century European algebraists. Of course, the operations with fractions can also be carried out in decimal form.

The disappointing feature of the decimal representation of fractions is that some simple fractions cannot be represented as decimals with a finite number of digits. Thus when we seek to express image as a decimal, we find that neither 0.3, nor 0.33, nor 0.333, and so on, suffices. All one can say in this and similar cases is that by carrying more and more decimal digits one comes closer and closer to the fraction, but no finite number of digits will ever be the exact answer. This fact is expressed by the notation

image

where the dots indicate that we must keep on adding threes to approach the fraction image more and more closely.

From the standpoint of applications the fact that some fractions cannot be expressed as decimals with a finite number of digits does not matter because we can always carry enough digits to obtain an answer as accurate as the application requires.

#### EXERCISES

1. What is the principle of positional notation?

2. Why is the number zero almost indispensable in the system of positional notation?

3. What is the meaning of the statement that zero is a number?

4. What two methods are there of representing fractions?

5. What principle determines the definitions of the operations with fractions?

4–3 IRRATIONAL NUMBERS

The Pythagoreans, as we noted earlier, were the first to appreciate the very concept of number, and sought to employ numbers to describe the basic phenomena of the physical and social worlds. Numbers to the Pythagoreans were also interesting in and for themselves. Thus they liked square numbers, that is, numbers such as 4, 9, 16, 25, 36, and so on, and observed that the sums of certain pairs of square numbers, or perfect squares, are also square numbers. For example, 9 + 16 = 25, 25 + 144= 169, and 36 + 64= 100. These relationships can also be written as

32 + 42 = 52, 52 + 122 = 132, and 62 + 82 = 102.

The three numbers whose squares furnish such equalities are today called Pythagorean triples. Thus 3, 4, 5 constitute a Pythagorean triple because 32 + 42 = 52.

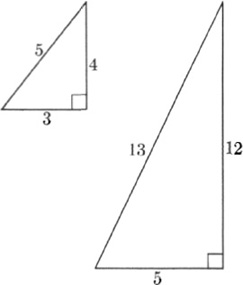


Fig. 4–1

The Pythagoreans liked these triples so much because, among other features, they have an interesting geometrical interpretation. If the two smaller numbers are the lengths of the sides or arms of a right triangle, then the third one is the length of the hypotenuse ([Fig. 4–1](#fig4_1)). Just how the Pythagoreans knew this geometrical fact is not clear, but assert it they did. They also claimed that in any right triangle, the square of the length of one arm added to the square of the length of the other gives the square of the length of the hypotenuse. This more general assertion is still called the Pythagorean theorem and a proof of it, such as we learn in high-school geometry, was given about 200 years later by Euclid. Pythagoras is said to have been so overjoyed with this theorem that he sacrificed an ox to celebrate its discovery.

This theorem proved to be the undoing of a central doctrine in the Pythagorean philosophy and caused woe and misery to many mathematicians. But before we pursue this story, we should look into a few simple properties of the whole numbers which are embodied in the following exercises.

#### EXERCISES

1. Prove that the square of any even number is an even number. [Suggestion: By definition every even number contains 2 as a factor.]

2. Prove that the square of any odd number is an odd number. [Suggestion: Every odd number ends in 1, 3, 5, 7, or 9.]

3. Let a stand for a whole number. Prove that if a2 is even, then a is even. [Suggestion: Use the result in Exercise 2.]

4. Establish the truth or falsity of the assertion that the sum of any two square numbers is a square number.

There are tragedies in mathematics also, and one of these struck the very group of mathematicians who deserved a better fate. The Pythagoreans had constructed, at least to their own satisfaction, a philosophy which asserted that all natural phenomena and all social and ethical concepts were in essence just whole numbers or relationships among whole numbers. But one day it occurred to a member of the group to examine the seemingly simplest case of the Pythagorean theorem. Suppose each arm of a right triangle ([Fig. 4–2](#fig4_2)) is 1 unit in length; how long, he asked, is the hypotenuse? The Pythagorean theorem says that the square of (the length of) the hypotenuse equals the sum of the squares of the arms. Hence if we call c the unknown length of the hypotenuse, then the theorem says that

c2 = 12 + 12

or

c2 = 2.

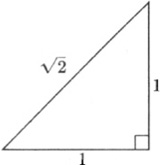


Fig. 4–2

Now 2 is not a square number, that is, a perfect square, and so c is not a whole number. But it certainly seemed reasonable to this Pythagorean that c should be a fraction; that is, there should be a fraction whose square is 2. Even the simple fraction image comes close to being the correct value because image, and this is almost 2. However, simple trial does not easily yield a fraction whose square is 2. Hence this Pythagorean became worried, and he decided to investigate the question of whether there is a fraction whose square is 2. We shall examine his reasoning which, as far as we know, is the same as that given in Euclid’s famous work on geometry, the Elements.

The number c which we seek to determine is one whose square is 2. Let us denote it by image. All we mean by this symbol is that it represents a number whose square is 2. And now let us suppose that image is a fraction a/b, where a and b are whole numbers. Moreover, to make matters simpler, let us suppose that any factors common to a and b are cancelled. Thus if a/b were image, for example, we would cancel the common factor 2 and write it as image. Hence we have assumed so far that

image

and that a and b have no common factors.

If [equation (1)](#eq_4_1) is correct, then by squaring both sides, a step which utilizes the axiom that equals multiplied by equals give equals (because we multiply the left side by image and the right side by a/b), we obtain

image

Again by employing the axiom that equals multiplied by equals yield equals, we may multiply both sides of this last equation by b2 and write

image

The left side of this equation is an even number because it contains 2 as a factor. Hence the right side must also be an even number. But if a2 is even, then, according to Exercise 3 above, a must be even. If a is even, it must contain 2 as a factor. That is, a = 2d, where d is some whole number. If we substitute this value of a in [(2)](#eq_4_1) we obtain

image

Since then

2b2 = 4d2,

we may divide both sides of this equation by 2 and obtain

image

We now see that b2 is an even number and so, by again appealing to the result in Exercise 3, we find that b is an even number.

What we have shown in the above argument is that if image, then a and b must be even numbers. But at the very outset we had cancelled any common factors in a and b; yet we find that a and b still contain 2 as a common factor. This result contradicts the fact that a and b have no common factors.

Why do we arrive at a contradiction? Since our reasoning is correct, the only possibility is that the assumption that image equals a fraction is not correct. In other words, image cannot be a ratio of two whole numbers.

This proof is so neat that one can almost believe the legend that Pythagoras sacrificed an ox in honor of its creation. But there are at least two reasons for discrediting this tale. The first is that if all the legends telling of Pythagoras sacrificing an ox were true, he could not have had time for mathematics. The second reason is that the above proof was not a triumph for the Pythagoreans but a disaster. The symbol image is a number because it represents the length of a line, namely the hypotenuse of the triangle in [Fig. 4–2](#fig4_2). But this number is not a whole number or a fraction. The Pythagoreans had, however, developed an embracing philosophy which asserted that everything in the universe reduced to whole numbers. Clearly, then, this philosophy was inadequate. Indeed the existence of numbers such as image was such a serious threat to the Pythagorean philosophy that another legend, more credible, states that the Pythagoreans, who were at sea when the above discovery was made, threw overboard the member who made it, and pledged to keep the discovery secret.

But secrets will out, and later Greeks not only learned that image is neither a whole number nor a fraction, but they discovered that there is an indefinitely large collection of other numbers which are not whole numbers or fractions. Thus image, image, image, and, more generally, the square root of any number which is not a perfect square, the cube root of any number which is not a perfect cube, and so on, are numbers which are not whole numbers or fractions. The number π, which is the ratio of the circumference of any circle to its diameter, is also neither a whole number nor a fraction. All these new numbers are called irrational numbers, the word “irrational” now meaning that these numbers cannot be expressed as ratios of whole numbers, although in Pythagorean times it meant unmentionable or unknowable.

If these irrational numbers are really so common and represent lengths of sides of triangles and circumferences of circles in terms of the diameters, why weren’t they encountered before? Didn’t the Babylonians and Egyptians run across them? They did. But since they were concerned only with having numbers serve their practical purposes, they used convenient approximations. Thus, when they encountered a length such as image, they were content to use a value such as 1.4 or 1.41. For π, as we noted in an earlier chapter, they used values even as crude as 3. Not only did these peoples use such approximations, but they never realized that the most complicated fraction or decimal could never represent an irrational number exactly. The Egyptians and Babylonians treated irrational numbers and their mathematics in general rather lightheart-edly. We may hail their blithe spirits, but mathematicians they never were.

The Greeks, as we know, were of a different intellectual breed and could not be content with approximations, but they also exhibited a weakness. Although they recognized that quantities exist which are neither whole numbers nor fractions, they were so convinced that the concept of number could not comprise anything else than whole numbers or fractions that they did not accept irrationals as numbers. Instead they thought of such quantities only as geometrical lengths or areas. Thus the Greeks never did develop an arithmetic of irrational numbers. In their astronomical work, for example, they used only whole numbers and fractions. The difficulty which the Greeks experienced also baffled all mathematicians up to modern times. The greatest mathematicians refused to accept irrationals as numbers and followed the Greek procedure of thinking about such quantities as lengths or areas. All these people wished that the Pythagoreans had thrown all irrational numbers overboard rather than the man who discovered them.

But the needs of society often oblige even mathematicians to face unpleasantnesses. In the seventeenth century, science began to develop at an amazing rate, and science needs quantitative results. It may be nice to know that image is a certain length and that image · image is an area, but this knowledge does not suffice when one needs numerical results. And so finally mathematicians had to accept the fact that if they were to treat numerically all the quantities that arise in scientific work, they must handle irrational numbers as numbers. The mathematicians’ refusal over centuries to grant irrationals the status of numbers illustrates one of the surprising features of the history of mathematics. New ideas are often as unacceptable in this field as they are in politics, religion, and economics.

The situation, then, which must be faced squarely is that there are other numbers besides whole numbers and fractions. It is, of course, quite understandable that whole numbers and fractions should have been created and used first, for these numbers arise in the simplest physical situations man encounters. The irrational numbers on the other hand are not commonly encountered. Only the application of a theorem such as the Pythagorean theorem brings them to our attention, and even then one must go through a proof such as that examined above, to see that they are not whole numbers or fractions. But the fact that irrational numbers are late-comers does not mean that they are less acceptable or less genuine numbers. Just as we gradually add to our knowledge of the varieties of human beings and animals which exist in our physical world, so must we broaden our knowledge of the varieties of numbers and with true liberality accept these strangers on the same basis as the already familiar numbers.

However, if we are to use irrational numbers, we must know how to operate with them, that is, how to add, subtract, multiply, and divide them. We have already noted with whole numbers and fractions that if we wish the operations to fit experience, we must formulate the operations accordingly. So it is with the irrational numbers. We could define addition, multiplication, and the other operations as we please. But if we wish these operations to represent physical situations, we must define them properly. However, there is no real difficulty here. Since irrational numbers are quantities, as are whole numbers and fractions, we may use the latter as a guide to the proper operations with irrational numbers.

Let us consider a few examples which will be sufficient to indicate the general principles. Should we say that

image

To answer this question let us consider the analogous question: May we say that

image

It is clear in the latter case that 2 + 3 does not equal image, for image is certainly less than 4. Hence we should not add the radicands, that is, the 2 and the 3, in the preceding equation. One might then ask, How much is image? Since both summands are numbers, the sum is also a number, but it cannot be written more compactly than image. This inability to combine the summands is not something new or troublesome. When we add 2 and image, for example, the answer continues to be image. We usually omit the plus sign and write image, but the summands are really not combined.

Let us consider next whether

image

Here too we shall see what the analogous operation with whole numbers suggests. Is it true that

image

The answer is clearly yes, and so we shall agree that to multiply square roots we shall multiply the radicands. That is,

image

The definitions of the operations of subtraction and division are also readily determined. Thus image yields a definite number, but the difference cannot be written any more compactly than image.

For division, say image, the procedure, as in the case of multiplication, is suggested by observing that

image

for this equation simply says that image. Hence we shall agree that

image

The general principle which these examples illustrate is that operations with irrational numbers are defined so as to agree with the same operations on whole numbers when the latter are expressed as roots. We could state our definitions in general form, but there is no need to do so.

In applications we often approximate irrational numbers by fractions or decimals because actual physical objects cannot be constructed exactly anyway. Thus if we had to construct a length which strictly should be image, we would approximate image. Since (1.4)2 = 1.96 and 1.96 is nearly 2, we could approximate image by 1.4. If we desired a more accurate approximation, we might determine to the nearest hundredth the number whose square approximates 2. Thus, since

(1.41)2 = 1.988 and (1.42)2 = 2.016,

we see that 1.41 is a good two-decimal approximation of image. We could, of course, improve still more on the accuracy of the approximation. We should, however, realize that no matter how many decimal places we employed, we would never obtain a number which is exactly image because any decimal with a finite number of digits or a whole number plus such a decimal is just another way of writing a fraction, whereas image, as the above proof showed, can never equal a quotient of two whole numbers.

The fact that we often approximate an irrational number when we wish to construct something raises a question which merits an answer. The question is, Why don’t we approximate irrational numbers wherever they arise and forget about operations with irrationals as such? For example, to calculate image, we could approximate image by, say 1.41, approximate image by 1.73, and then multiply 1.41 by 1.73. The answer is 2.44, and since (2.44)2 is 5.95, we see that we have a good approximation to image. If we wanted a more accurate answer, we could approximate image and image more closely and then multiply. One reason we do not approximate in mathematics proper is that mathematics is an exact science. It insists on reasoning as rigorous as human beings can perform. We pay a price for this rigor by expending more thought and effort, but we shall see that mathematics has made its contributions just because it insists on exactness.

There is also a practical advantage in working with irrational numbers as such. Let us suppose that some problem required us to calculate image, that is, image. The person who insists on approximating would now approximate image to some number of decimal places, for example, 1.732, and then calculate (1.732)4 While the practical person takes an hour to calculate and check his arithmetic, the mathematician would see at once that

image

and could spend the rest of the hour in refreshing sleep. Moreover, the mathematician’s answer is exact, whereas the practical man’s answer is not accurate even to the four demical places with which he started, because the product of two approximate numbers is less accurate than either factor. To achieve an answer accurate to four decimal places, the practical man would have to use an approximation of image containing seven decimal places and then multiply.

The irrational number is the first of many sophisticated ideas which the mathematician has introduced to think about and cope with the real world. The mathematician creates these concepts, devises ways of working with them which fit real situations, and then uses his abstractions to think about the phenomena to which the ideas apply.

#### EXERCISES

1. Express the answers to the following problems as compactly as you can:

a) image

b) image

c) image

d) image

e) image

f) image

g) image

h) image

i) image

j) image

k) image

2. Simplify the following:

a) image

b) image

c) image

[Suggestion: image]

3. Criticize the following argument: No irrational number can be expressed as a decimal with a finite number of decimal places. The number image cannot be expressed as a decimal with a finite number of decimal places. Hence image is an irrational number.

4–4 NEGATIVE NUMBERS

One more addition to the number system which has considerably extended the power of mathematics comes from far-off India. Numbers are commonly used to represent an amount of money, in particular the amount of money which a person owns. Perhaps because the Hindus were in debt more often than not, it occurred to them that it would also be useful to have numbers which represent the amount of money one owes. They therefore invented what are now called negative numbers, while the previously known numbers are called positive numbers. Thus numbers which we denote by −3, −image, and −image came into existence. Where necessary to distinguish clearly positive from negative numbers or to emphasize what is positive as opposed to what is negative, one writes +3 or + image instead of 3 or image.

It is not necessary, incidentally, to use such symbols as −3 to represent the negative counterpart to 3. Modern banks and large commercial corporations, which deal with negative numbers continually, often write these in red ink, whereas positive numbers are written in black ink. However, we shall find that placing the minus sign in front of a number to indicate a negative number is a convenience.

The use of positive and negative numbers is not limited to the representation of assets and debts. One represents temperatures below 0° as negative temperatures, while temperatures above 0° are positive. Likewise heights above and below sea level can be represented by positive and negative numbers, respectively. It is sometimes convenient to represent time after and before a specified event by positive and negative numbers. For example, using the birth of Christ as the event, the year 50 B.C. can just as well be described as the year −50.

To derive more use from the concept of negative numbers it must be possible to operate with them just as we operate with positive numbers. The operations with negative numbers and with negative and positive numbers together are easy to understand if one keeps in mind the physical significance of these operations. For example, suppose a man has assets of 3 dollars and debts of 8 dollars. What is his net wealth? Clearly the man is 5 dollars in debt. The same calculation is represented in terms of positive and negative numbers by stating that the amount 8 dollars must be taken from 3 dollars, that is, 3 − 8, or that a debt of 8 dollars must be added to assets of 3 dollars, that is, + 3 + (− 8). The answer is obtained by subtracting the smaller numerical value (that is, the smaller number without regard to sign) from the larger numerical value and giving the answer the sign attached to the larger numerical value. That is, we subtract 3 from 8 and call the answer negative because the larger numerical value, namely 8, has the minus sign attached to it.

Since negative numbers represent debts and subtraction usually has the physical meaning of “taking away” or “removing,” then the subtraction of a negative number means the removal of a debt. Thus, if a person has assets of, say 3 dollars, but this figure already takes into account a debt of 8 dollars, the removal or cancellation of the debt leaves the person with assets of 11 dollars. Mathematically we say +3 − (− 8) = + 11. In words, to subtract a negative number we add the corresponding positive number.

Suppose a man goes into debt at the rate of 5 dollars per day. Then in 3 days after a given date he will be 15 dollars in debt. If we denote a debt of 5 dollars as −5, then going into debt at the rate of 5 dollars per day for 3 days can be stated mathematically as 3 · (−5) = −15. That is, the multiplication of a positive and a negative number yields a negative number whose numerical value is the product of the two given numerical values.

In the very same situation in which a man goes into debt at the rate of 5 dollars per day, his assets three days before a given date are 15 dollars more than they are at the given date. If we represent time before the given or zero date by −3 and the loss per day as −5, then his relative financial position 3 days ago can be expressed as − 3 · (− 5) = + 15; that is, to consider his assets three days ago, we would multiply the debt per day by −3, whereas to calculate the financial status three days in the future, we multiply by +3. Hence the result is +15 in the former case compared to − 15 in the latter.

There is one more definition concerning negative numbers which is readily seen to be sensible. For the positive numbers and zero we say for obvious reasons that 3 is greater than 2, that 2 is greater than image, and that any positive number is greater than zero. The negative numbers are said to be less than the positive numbers and zero. Moreover, we say that −5 is less than −3, or that − 3 is greater than −5. If one thinks of these various numbers as representing people’s wealth, then the agreement concerning their order fits our usual understanding of relative wealth. A person whose financial status is −3 is wealthier than one whose status is −5; one is better off to be 3 dollars than 5 dollars in debt. Incidentally, the symbol > is used to denote “greater than” as in 5 > 3, and the symbol < denotes “less than” as in −5 < −3.

The relative position of the various positive and negative numbers and zero is readily remembered if one visualizes these numbers as points on a line as shown in [Fig. 4–3](#fig4_3). The figure is really not different from that obtained by moving a thermometer scale into a horizontal position.

image

Fig. 4–3

The above situations, which illustrate how the definitions of the operations with positive and negative numbers were suggested, are of course by no means the only ones in which positive and negative numbers are employed. Indeed the usefulness of negative numbers would hardly be great were this the case. However, these simple financial transactions show not only how mathematicians arrived at the definitions, but that there is no more mystery about negative numbers than about positive ones. The definitions represent in abstract form what takes place physically, and, as with all numbers, we can think in terms of the abstractions to arrive at a knowledge of physical happenings.

It may be of some comfort to the reader to know that the concept of negative numbers, like the concept of irrational numbers, was resisted by mathematicians for several hundred years. The history of mathematics illustrates the rather significant observation that it is more difficult to get a truth accepted than to discover it. The mathematicians to whom “number” meant whole numbers and fractions found it hard to accept negative numbers as true numbers. They, too, failed to realize for centuries that mathematical concepts are man-made abstractions which can be introduced at will if they can serve useful purposes.

#### EXERCISES

1. Suppose a man has $3 and incurs a debt of $5. What is his net worth?

2. Suppose a man owes $5 and then incurs a new debt of $8. Use negative numbers to calculate his financial condition.

3. Suppose a man owes $5 and earns $8. Use positive and negative numbers to calculate his net worth.

4. Suppose a man owes $13, and a debt of $8 is cancelled. Use negative numbers to calculate his net worth.

5. A man loses money in business at the rate of $100 per week. Let us denote this change in his assets by − 100 and let us denote time in the future by positive numbers and time in the past by negative numbers. How much will the man lose in 5 weeks? How much more was the man worth 5 weeks ago?

4–5 THE AXIOMS CONCERNING NUMBERS

In the preceding chapter we said that mathematics proceeds by deductive reasoning from explicitly stated axioms. Yet thus far in this chapter we have said nothing about axioms. The reason is simply that the axioms concerning numbers are such obvious properties that we use them automatically without realizing that we are doing so.

This situation may perhaps be better understood by means of an analogy. Whenever a child at play throws a ball up into the air, he expects that the ball will come down. He is really assuming that all balls thrown up will come down. Of course, this assumption is well founded in experience; nevertheless, the child’s expectation that the ball will come down is a deduction from the assumption just stated and the additional premise that he is throwing a ball up into the air. Recognition of the fact that he has made an assumption makes clear the reasoning, conscious or unconscious, behind the act.

To understand the deductive process in the mathematics of numbers, as well as in geometry, we must recognize the existence and use of the axioms. We do not hesitate to say that 275 + 384= 384 + 275. Surely we did not add 384 objects to 275, count the total, then add 275 objects to 384, count that total, and check that the two totals agree. Rather, whenever in our experience we combined two groups of objects, we found that we obtained the same total collection regardless of whether we put the first group with the second or the second with the first. Of course, our evidence to the effect that the order of addition is immaterial is limited to a small number of cases, whereas in practice we use this fact with all numbers. Hence, we are really making an assumption, namely, that for any two numbers a and b, integral, fractional, irrational, and negative, the order of addition will not affect the result. Thus our assumption also includes the affirmation that image. It is important for another reason to recognize that this assumption is being made. Numbers are not apples or cows. They are abstractions from physical situations. Mathematics works with these abstractions in order to deduce information about physical situations. However, if the axioms are not well chosen, the deductions will not apply. Hence it is well to note what assumptions are being employed and to ascertain that they are well founded in experience.

Let us, therefore, note the axioms which we have been using and will continue to use. The first axiom is the one discussed in the preceding paragraph:

AXIOM 1. For any two numbers a and b,

a + b = b + a.

The axiom is called the commutative axiom of addition because it says that we can commute or interchange the order of the two numbers to be added. We note that subtraction is not commutative, that is, 3 − 5 does not equal 5 − 3.

If we had to calculate 3 + 4 + 5, we could first add 4 to 3 and then add 5 to this result, or we could add 5 to 4, and then add this result to 3. Of course, the result is the same in the two cases, and this is exactly what our second axiom says.

AXIOM 2. For any numbers a, b, and c,

(a + b) + c = a + (b + c).

This axiom is called the associative axiom of addition because we can associate the three numbers in two different ways in performing the addition.

The two axioms we have just discussed have their analogues for the operation of multiplication.

AXIOM 3. For any two numbers a and b,

a · b = b · a.

This axiom is called the commutative axiom of multiplication. Incidentally, the dot which is used to denote multiplication is omitted if there is no danger of misunderstanding. Thus, we could as well write ab = ba. The axiom is clearly a property of numbers; yet we sometimes fail to recognize that it is applicable. Many a student hesitates to write 5 · a instead of a · 5. But the commutative axiom says that the two expressions are equal. We might note in this context that the operation of division is not commutative, for 4 ÷ 2 does not equal 2 ÷ 4.

AXIOM 4. For any three numbers a, b, and c,

(ab)c = a(bc).

This axiom is called the associative axiom of multiplication. Thus (3·4)5 = 3(4·5).

We also find in our work with numbers that it is convenient to use the number 0. To recognize formally that there is such a number and that it has the properties which its physical meaning requires, we state another axiom.

AXIOM 5. There is a unique number 0 such that

a) 0 + a = a for every number a,

b) 0 · a = 0 for every number a,

c) if ab = 0 then either a = 0 or b = 0 or both are 0.

The number 1 is another whose properties are somewhat special. Again, we know from the physical meaning of 1 just what its properties are, but if one is to justify the operations with 1 by appealing to axioms rather than to physical meaning, there must be a statement which tells us just what these properties are. In the case of 1, it is sufficient to specify a sixth axiom.

AXIOM 6. There is a unique number 1 such that

1 · a = a

for every number a.

In addition to adding and multiplying any two numbers, we also have physical uses for the operations of subtraction and division. We know that given any two numbers a and b, there is a number c which results when b is subtracted from a. From a practical standpoint, it is helpful to recognize that subtraction is the inverse operation to addition. What this means is simply that if we have to find the answer to 5 − 3 we can and, in fact, do ask ourselves what number added to 3 gives 5. If we know addition, we can then answer the subtraction problem. Even if we obtain the answer by a special subtraction process, and we do in the case of large numbers, we check it by adding the result to what we subtracted to see if it gives the original number or minuend. Hence a subtraction problem such as 5 − 3 = x really asks for what number x added to 3 gives 5, that is, x + 3 = 5.

In our logical development of the number system we wish to affirm that we can subtract any number from any other number and we phrase this statement so that the meaning of subtraction is precisely what it is, the inverse of addition.

AXIOM 7. If a and b are any two numbers, there is a unique number x such that

a = b + x.

Of course, the quantity x is what we usually denote by a − b.

The relation of division to multiplication is also that of an inverse operation. When we seek the answer to image we may happen to know directly from experience that the answer is 4. But if we don’t, we can reduce the division problem to a multiplication problem and ask what number, x, multiplied by 2 gives 8, and if we know multiplication we can find the answer. Here too, as in the case of subtraction, even if we use a special division process such as long division to find the answer, we check the answer by multiplying the divisor by the quotient to see if the product is the dividend. The reason for doing this is simply that the basic meaning of a/b is to find some number x such that bx = a.

In our logical development of the number system we affirm that we can divide any number by any other number (except 0) and we phrase the assertion so that the meaning of division is precisely what it actually is, the inverse of multiplication.

AXIOM 8. If a and b are any two numbers, except that b ≠ 0, then there is a unique number x such that

bx = a.

Of course x is the number usually denoted by a/b.

The next axiom is not quite so obvious. It says, for example, that 3·6 + 3·5 = 3(6 + 5). In this example we can perform the calculation to see that the left and right sides are equal, but this is really not necessary. Suppose we had 157 cows in one herd and 379 in another, and each herd increased sevenfold. The total number of cattle is then 7 · 157 + 7 · 379. But if the original two herds were one herd with 157 + 379 cows, and this single herd increased sevenfold, we would have 7(157 + 379) cows. It is physically clear that we have the same number of cows now as before, that is, that 7 · 157 + 7 · 379 = 7(157 + 379). Stated in general terms, the axiom is:

AXIOM 9. For any three numbers a, b, and c,

ab + ac = a(b + c).

This axiom, called the distributive axiom, is very useful. For example, to calculate 571 · 36 + 571 · 64 we can apply the axiom to state that this quantity is 571(36 + 64) or 571 · 100 or 57,100. We say often that we have factored the quantity 571 out of the sum 571 · 36 + 571 · 64.

We should note that from

ab + ac = a(b + c)

we can also state that

ba + ca = (b + c)a,

because in each term of the first equation we can apply the commutative axiom of multiplication to change the order of the factors.

We often use the second form of the distributive axiom. Thus, suppose a is some number and we wish to calculate 5a + 7a. We can replace this sum by (5 + 7) a, and obtain 12a.

The distributive axiom is also applicable in the following situation. Suppose we have to calculate

image

One might be tempted to cancel the two numbers 296. But this is incorrect. The given fraction means

image

and the distributive axiom tells us that we may write instead

image

In addition to the above axioms, we have the following evident properties of numbers:

AXIOM 10. Quantities equal to the same quantity are equal to each other.

AXIOM 11. If equal quantities are added to, subtracted from, multiplied with, or divided into equal quantities, the results are equal. However, division by zero is not permitted.

The set of axioms we have just given is not complete; that is, it does not form the logical basis for all of the properties of the positive and negative whole numbers, fractions, and irrational numbers. However, the set does provide the logical basis for what is usually done with numbers in ordinary algebra. Moreover, it does give some idea of what the axiomatic basis for mathematical work with numbers amounts to.

Now that we have the axioms, what do we do with them? We can prove theorems about numbers. Let us consider a few examples. Negative numbers were introduced to represent physical happenings such as debts or time before a given event. When we examined the physical situation in which we wished to use these numbers, we found that if the numbers are to be useful then we should agree that, for example,

– 2 · 3 = –6 and –2 · (–3) = 6.

There we agreed to operate with positive and negative numbers so as to make the results fit the physical situation. In the deductive approach to numbers we prove on the basis of our axioms that certain theorems are correct. Let us prove that a positive number times a negative number is negative.

Let a and b be positive numbers. Then − b is a negative number; for example, −b = −5. We shall prove that a(− b) = −ab. We know by Axiom 7, wherein we let a be 0, that if b and 0 are given, then there is a number x such that b + x = 0. This number x is denoted by 0 − b or −b. Then

b + (–b) = 0.

Now we can multiply both sides of this equation by a, and since equals multiplied by equals give equals, we have

a[b + (–b)] = a · 0.

Now a · 0 = 0 · a by [Axiom 3](#ch4axi3), and 0 · a = 0 by [Axiom 5](#ch4axi5). By applying the distributive axiom, [Axiom 9](#ch4axi9), to the left-hand side of our equation, we obtain

image

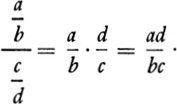
and now we see that a(– b) is the number which added to ab gives 0. But [Axiom 7](#ch4axi7) says that given ab and 0 (these are the a and b of [Axiom 7](#ch4axi7)), there is a unique number x such that

image

This number x is denoted by 0 − ab or −ab. But [equation (1)](#eq_4_1a) says that a(–b) is the number which added to ab gives 0. Since there is just one such number which when added to ab gives 0, and that number we know is –ab, it must be be that a(–b) = –ab.

The proof is now complete and yet may not be convincing. The reason is simply that we are so accustomed to operating with numbers on the basis of physical arguments and experience with them that we have not accustomed ourselves to reasoning with numbers on an axiomatic basis.

Let us consider another proof. In [Section 4–2](#ch4_2) we gave a physical argument to show that if we divide a/b by c/d then we can get the answer by inverting c/d to obtain d/c and then multiplying; that is,



We can prove, on the basis of our axioms, that this rule of inverting the denominator and then multiplying is correct.

To divide a/b by c/d is to find a number x such that

image

Now [Axiom 8](#ch4axi8) tells us that there is a unique number x which satisfies such an equation. We do know that

image

because if we cancel common factors in the numerator and denominator of the right side we obtain a/b. Hence one number which can serve as the x which satisfies [equation (3)](#eq_4_3a) is ad/bc. But since the value of x is unique, x = ad/bc. Thus the result of dividing a/b by c/d is ad/bc. We note that the answer ad/bc is obtained by multiplying a/b by the inverse of c/d. Hence to divide one fraction by another, we invert the denominator and multiply.

This proof, like the preceding one, may not be convincing, and the reason is the same. We are not accustomed to reasoning about numbers on an axiomatic basis. Rather, we have relied upon the physical meaning of numbers and operations. Historically, the mathematicians did the same thing. They learned to operate with numbers by noting the uses to which numbers were put, and they constructed the axiomatic basis long afterward, just to satisfy themselves that deductive proofs of the properties of numbers could be made.

Since we, too, are accustomed to the properties and operations with numbers since childhood, and we are sure of these properties, we shall not often cite axioms to justify our steps. Thus if we write 3a in place of a · 3, we shall not cite the commutative axiom of multiplication as the justification for this step. In fact, it would be pedantic to do so. The axioms are useful, rather, in helping us to determine what is correct when our experience fails us or leaves us in doubt. However, we should not lose sight of the fact that the mathematics built upon the number system is a deductive system. This point needs emphasis because we begin to learn arithmetic at an early age by rote and thereafter we tend to operate with numbers mechanically without perceiving that we are constantly using axioms of numbers.

#### EXERCISES

1. Do you believe that

256(437 + 729) = 256 · 437 + 256 · 729?

Why?

2. Is it correct to assert that

a(b − c) = ab − ac?

[Suggestion: b − c = b + (−c).]

3. Perform the operations called for in the following examples:

a) 3a + 9a

b) a · 3 + a · 9

c) image

d) 7a − 9a

e) 3(2a + 4b)

f) (4a + 5b)7

g) a(a + b)

h) a(a − b)

i) 2(8a)

j) a(ab)

4. Carry out the multiplication:

(a + 3)(a + 2).

[Suggestion: Regard (a + 3) as a single quantity and apply the distributive axiom.]

5. Calculate (n + 1)(n + 1).

6. If 3x = 6, is x = 2? Why?

7. If 3x + 2 = 7, is 3x = 5? Why?

8. Is the equality x2 + xy = x(x + y) correct?

9. Is it correct to assert that

a + (bc) = (a + b)(a + c)?

image 4–6 APPLICATIONS OF THE NUMBER SYSTEM

Something of the power, methodology, and subtlety of mathematical reasoning can already be seen in the applications which have been made of the several types of number. Indeed, we shall see that these resulted in significant scientific discoveries.

Let us begin with some rather simple matters. Suppose a man drives a car for one mile at 60 miles per hour and for another mile at 120 miles per hour. What is his average speed? We tend to answer this question by applying the common procedure for finding an average. Thus, if a man buys one pair of shoes for $5 and another for $10, the average price is $5 + $10 divided by 2, or $7.50. Hence it would seem as though the average speed in the above problem should be 60 + 120 divided by 2, or 90 miles per hour. However, this answer is not correct. The number 90 is an average in the arithmetic sense, but it is not the average we seek. The average speed should be that speed which would enable the man to drive the two miles in the same time as it took him to drive that distance at the two different speeds. Now, it took the man 1 minute to drive the first mile and it took him image minute to drive the second mile. Hence it took him 1image minutes to drive 2 miles. We now ask, what average speed maintained for 1image minutes would cover 2 miles? Since the average speed multiplied by the total time should give the total distance, the average speed is the total distance divided by the total time, that is,

image

The average speed is then image miles per minute or 80 miles per hour.

The point of this example, not a momentous one to be sure, is merely that the unthinking, blind application of arithmetic does not produce the correct result. The notion of average speed serves a physical purpose, and unless we are clear about what average speed is supposed to mean, we shall not profit by the use of arithmetic.

#### EXERCISES

1. A man can row a boat in still water at 6 mi/hr. He plans to row upstream for 12 mi and then back in a river whose current flows at 2 mi/hr. Thus his speed upstream is 4 mi/hr and his speed downstream is 8mi/hr. He reasons that his average speed is 6 mi/hr and that the entire trip of 24 mi should therefore take 4 hr. Is this reasoning correct?

2. Suppose that a merchant sells apples at a price of 2 for 5¢ and oranges at 3 for 5¢. To make his arithmetic simpler he decides to sell any 5 pieces of fruit for 10¢ or at the average price of 2¢ per piece. Thus, if he sells 2 apples and 3 oranges, he sells 5 pieces of fruit at 2¢ each and receives the same 10¢ as if he had sold them at the original separate prices. Is the merchant’s average price correct? [Suggestion: Consider what results if he sells 12 apples and 12 oranges.]

3. Suppose the merchant wishes to sell a apples and b oranges to some customer at the prices given in Exercise 2. What should the average price be?

4. Given the data of Exercise 2, is there an average price which would be correct no matter how many apples and how many oranges are sold?

5. One man can dig a certain ditch in 2 days and another can dig the same ditch in 3 days. What is their average rate of ditch-digging per day?

Let us consider next an application of simple arithmetic to genetics. Suppose we have before us 2 red aces and 2 red kings from the usual deck of 52 cards. How many different pairs consisting of one ace and one king can be put together? Since each ace can be paired with either of 2 kings, there are 2 different pairs for any one ace. Since we have 2 aces, there are 2 · 2 or 4 different pairs.

Now let us suppose that we have 2 red aces, 2 red kings, and 2 red queens. How many different sets consisting of one ace, one king, and one queen can we form with the given cards? We saw above that there are 4 different pairs of aces and kings. With each of these 4 pairs we can place 2 different queens. Hence there are 4 · 2 or 8 different sets of 3 cards. We note that 4 · 2 = 2 · 2 · 2 = 23.

If we have 2 red aces, 2 red kings, 2 red queens, and 2 red jacks, the number of different sets, each consisting of one ace, one king, one queen, and one jack, can also be readily calculated. Each of the 8 choices of ace, king, and queen can be paired with each of the 2 jacks. Hence there are 8 · 2 or 16 choices in all. Now,

8 · 2 = 4 · 2 · 2 = 2 · 2 · 2 · 2 = 24.

Clearly, if we had 10 different pairs of cards and had to make all possible choices of 10 cards, one from each pair, the number of all possible sets of 10 cards would be

2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 = 210 = 1024.

This simple reasoning about cards has an important application to genetics. The reproductive cells (as well as ordinary cells) of the human male contain 24 pairs of chromosomes. When a sperm cell is formed from the reproductive cell, it contains 24 chromosomes, each coming from one of the 24 pairs. Hence a sperm cell can be formed in 224 possible combinations. The reproductive cells of the human female also contain 24 pairs of chromosomes. An ovum formed from the female reproductive cell contains 24 chromosomes, each coming from one of the 24 pairs of the reproductive cell. Hence there are 224 possible ways in which an ovum can be formed. In conception, any one sperm joins, or fertilizes, any one ovum. Since there are 224 possible sperms and 224 possible ova, the number of possible chromosome combinations for the fertilized ovum is then

224 · 224 = 16,777,216 · 16,777,216 = 281,474,976,710,656.

This is the number of possible variations in the genetic make-up of any one child a man and wife may have. Actually, the number of variations is somewhat larger. Each chromosome contains genes, and these determine the hereditary qualities. Biologists have found that any two paired chromosomes in a reproductive cell may exchange some genes, and this exchange gives rise to new varieties of sperm cells and ova.

#### EXERCISES

1. The usual deck of 52 cards contains 4 different aces and 4 different kings. How many different pairs of cards, each pair consisting of one ace and one king, can be formed from the aces and kings?

2. A manufacturer offers his automobile in 3 different colors, with or without a heater, and with or without a radio. How many different choices can a purchaser make?

3. A girl has 3 hats, 2 dresses, and 2 pairs of shoes. How many different costumes does she have?

4. There are six numbers on a die (singular of dice). How many different pairs of numbers can show up on a throw of a pair of dice? The two dice are to be marked so that a throw of a 2 on one die (say A) and of a 5 on the other (say B) can be distinguished from the reverse arrangement (5 on A and 2 on B).

We have already discussed the fact that our method of writing quantities uses the idea of positional notation in base ten (see [Section 4–2](#ch4_2)). However, some civilizations used other numbers as a base. For example, the Babylonians, for reasons that are obscure, selected 60. This system was taken over by the Greek astronomers and was used in Europe for many mathematical and all astronomical calculations as late as the seventeenth century. It still survives in our practice of dividing hours and angles into 60 minutes and 60 seconds. In adopting ten as a base, Europe followed the practice of the Hindus. Let us challenge history and see whether we can derive some advantage from a change to a new base.

We shall choose base six. The quantities from zero to five would be designated by the symbols 0, 1, 2, 3, 4, 5, as in base ten. The first essential difference comes up when we wish to denote six objects. Since six is to be the base, we would no longer use the special symbol 6, but place the 1 in a new position to denote 1 times the base, just as in base ten the 1 in 10 denotes one times the base, or the quantity ten. Hence, to write six in base six, we would write 10, but now the symbols 10 means 1 times six plus 0. Thus the symbols 10 can denote two different quantities, depending upon the base employed. Seven in base six would be written 11, because in base six these symbols mean 1 times six + 1, just as 11 in base ten means 1 times ten + 1. Again the symbols 11 represent different quantities, depending upon the base implied. As another example, to denote twenty-two in base six we write 34, because these symbols now mean 3 times six + 4.

In base ten, to write numbers larger than ninety-nine, we use a third position, the hundreds’ place, to indicate tens of tens. Similarly in base six, when we reach numbers larger than thirty-five, we use a third position to denote sixes of sixes. Thus thirty-eight would be written in base six as 102, wherein the one means 1 times six times six, the 0 means 0 times six, and the 2 denotes just 2 units. To express very large numbers we would use four-place numbers, five-place numbers, and so forth.

We can perform the usual arithmetic operations in base six. However, we would have to learn new addition and multiplication tables. For example, in base ten we write 5 + 3 = 8, whereas in base six, eight must be written 12. Hence our addition table would have to state that 5 + 3 = 12. Likewise, our new multiplication table would have to list 3 · 5 = 23, because fifteen is 2 · 6 + 3 or, in base six, 23. When we learned to use base ten, we had to memorize the result of adding each number from 0 to 9 to every number from 0 to 9, and the result of multiplying each number from 0 to 9 with every number from 0 to 9. For base six we would have to learn to add (and multiply) only numbers from 0 to 5 to (or with) the numbers of this set. Thus our addition and multiplication tables would be shorter, and we would learn arithmetic sooner as youngsters. We might even pass the hurdle of arithmetic so easily that we might get to like mathematics. The only disadvantage of base six would be that to represent large quantities we would have to use more digits. For example, the quantity fifty-four, written as 54 in base ten, must be written as 130 in base six, because fifty-four equals 1 · 62 + 3 · 6 + 0.

There are people who campaign for the adoption of base twelve, because it offers special advantages. For one thing, more fractions can be written as finite decimals in base twelve. Thus image must be written as the unending decimal 0.333 . . . in base ten, but can be written as 0.4 in base twelve, since in this base 0.4 means image. Also, since the English system of denoting length calls for 12 inches in one foot, we could, for example, express 3 feet and 6 inches as 36 inches in base twelve, whereas to express this number of inches in base ten, we must first calculate 3 · 12 + 6 and then write 42 in base ten. To a limited extent we could use base twelve in our method of recording time. In the United States the day has two sets of twelve hours, and in base twelve the hours of the day would run from 0 to 20. Whereas determining what 7 hours after 6 o’clock will be requires at present some computation, under the addition table for base twelve we would state at once that 7 + 6 = 11. However, base ten is now so widely used that a change to another base for ordinary daily use or commerce is hardly likely.

In the subject of bases we have an idea that was pursued for centuries, largely as an interesting and amusing speculation, but which suddenly became highly important in science and even in the commercial world. For several centuries mathematicians worked on the design of machines which would perform arithmetical computations quickly and thus remove a good deal of the drudgery of arithmetic. Although some types of computing machines were invented and used, mathematicians saw their golden opportunity in the electronic devices developed by modern radio engineers. The key is the radio vacuum tube, which can be made to pass current by applying a voltage to it or can be kept inactive. Two maneuvers are thus possible. In base two all numbers require only two symbols, the 0 and the 1. A typical number would be 1011, which means

1 · 23 + 0 · 22 + 1 · 2 + 1.

This number can be recorded by the machine by employing four tubes, one for the units’ place, another for multiples of 2, a third for multiples of 22, and a fourth for multiples of 23. To record 1011, the first, second, and fourth tubes can be activated, and the third, which records the third place in the number, kept inactive. The currents passed by the tubes which “fire” are recorded by the machine in special circuits. Another number can then be fed into the machine. Let us suppose that it is to be added to the first number. The result of having two l’s in the same place means, in base 2, that the sum is to be 0 and a 1 carried over to the next place. This operation is readily performed by the circuits. While this description of an electronic computer certainly doesn’t begin to present the ingenious ideas which engineers and mathematicians have incorporated, we may perhaps see that the workings of a vacuum tube are ideally suited to operations in base two.

To take advantage of the fact that computers can perform calculations in base two, the numbers to be worked on are converted beforehand from base ten to base two and then fed to the machine along with other instructions. The machine then operates in this latter base. The result is, of course, reconverted to base ten.

Because computers work with microsecond speed, they are exceedingly valuable in any commercial or industrial organization which must process a lot of numerical data. Calculations in banks, insurance companies, and in industry, which used to require an immense amount of human labor, are now performed by machines. Computers are the first in a new series of machines which keep track of great quantities of data, select information from millions of cards on which data are recorded, plan factory operations, direct machinery, and may soon provide translations of foreign-language publications.

Electronic computing machines are an enormous boon to science and mathematics also. The arithmetic required to extract concrete information from mathematical formulas is often so lengthy that it would take years to perform these calculations. Computing machines do such work in hours. Moreover, mathematicians no longer hesitate to work on problems which will lead to extensive computations, because they now know that their work will not be in vain.

Computing machines may help us to learn more about the action of the human brain. According to biologists, the nerve cells in the nerve chains and the cells in our brains respond to electrical impulses much as a vacuum tube does. Just as a tube will “fire” when it receives electrical current beyond a certain minimum value and remain inactive otherwise, so do the nerve cells in the nerve chains transmit an electrical impulse to whatever organ they may lead when this impulse exceeds a threshhold value; otherwise they are inactive. Computing machines also have a memory; that is, partial results of calculations are stored automatically in a special device, called the memory. When these results are needed, the memory device releases them. Thus the result of an addition process might be stored in the memory device until the result of some multiplication process is obtained and then, if so instructed, the machine will add these two results. The machine’s memory device, then, functions somewhat like the human memory. Hence, in two respects at least, electronic computing machines simulate the actions of human nerves and memory. Though machines are, in speed, accuracy, and endurance, superior to the human brain, one should not infer, as many popular writers are now suggesting, that machines will ultimately replace brains. Machines do not think. They perform the calculations which they are directed to perform by people who have the brains to know what calculations are wanted. Nevertheless, we undoubtedly have in the machine a useful model for the study of some functions of the human brain and nerves.

#### EXERCISES

1. Construct an addition table for base six.

2. Construct a multiplication table for base six.

3. Construct an addition and multiplication table for base two.

4. The following numbers are in base ten:

9, 10, 12, 36, 48, 100.

Write the respective quantities in base six.

5. The following numbers are in base six:

5, 10, 12, 20, 100.

Write the respective quantities in base ten.

6. Write the fraction image as a decimal in base six.

7. The quantity 0.2 is in base six. Write the corresponding quantity in base ten.

8. The number 101 is written in some unknown base and equals ten. What is the base?

9. Find the least number of weights needed to weigh, to the nearest pound, objects weighing from 0 to 63 lb. The scale to be used contains two pans and the weights are to be put in one pan. [Suggestion: Consider the problem of representing all numbers from 0 to 63 in base 2.]

Some of the most remarkable uses of numbers, which have led to profound discoveries, are found in the study of the structure of matter. During the early and middle part of the nineteenth century, certain basic experimental facts about the varieties of matter found in nature led, after some inevitable fumbling on the part of John Dalton, Amadeo Avogadro, and Stanislav Cannizzaro, to the theory that all matter is made up of atoms. Thus hydrogen, oxygen, chlorine, copper, aluminum, gold, silver, and all other varieties of matter are composed of atoms. Experimental techniques were developed to measure the relative weights of the atoms of different elements. The convenient unit of weight chosen was image the weight of the oxygen atom, so that the weight of the atom of oxygen is 16. The weight of the hydrogen atom then proved to be 1.0080, that of copper 63.54, that of gold 197.0, and so on. By this time, too, a number of chemical properties of these various elements, such as their melting temperatures, boiling temperatures, and their ability to combine with other elements to form compounds, had been determined.

The question which had begun to stir the chemists was whether there existed any law or principle which utilized the atomic weights of these elements. The crowning discovery was made in 1869 by Dimitri Ivanovich Mendeléev (1834–1907). He found, as he began to arrange the elements in order of increasing atomic weight, that every eighth element, starting from a given one, had chemical properties similar to the first one. Thus, the gases fluorine and chlorine, the latter the eighth element starting from fluorine, both combine readily with metals. However, as he continued to place each of the 63 different elements known in his time in the eighth position after the one with similar chemical properties, he saw that he had to leave blank spaces. Mendeléev was so much impressed with the periodicity of chemical properties that he did not hesitate to leave the blank spaces and to affirm that there must be elements to fill these spaces. Since each of these missing elements should have chemical properties similar to those of the element found in the eighth position preceding or succeeding the missing element, he could even predict some of the properties of the unknown elements. Mendeléev described the properties of three of the missing elements, and his immediate successors discovered them. These are now called scandium, gallium, and germanium. Still later, others were found. The interesting fact about Mendeléev’s work from the mathematical standpoint is that he had no physical explanation of why elements eight positions removed from one another should have similar chemical properties. He knew only that the number eight was the key to the arrangement, and he followed this mathematical guide faithfully. Long after Mendeléev’s time, other elements, for example helium, were found, which do not fit into this arrangement, but his periodic table is still the basic one which all students of chemistry learn today.

Simple arithmetic continued to play a leading role in subsequent developments of atomic theory. The continuing study of the atomic weights of various elements and their chemical properties showed that elements formerly regarded as pure were really not so. Thus, there are two kinds of hydrogen. These have similar chemical properties, but different atomic weights; in fact, one is twice as heavy as the other. Since both were previously called hydrogen, and since they do, in any case, have similar chemical properties, these two forms of hydrogen are called isotopes of hydrogen. Likewise, there is not one substance, oxygen, but there are three, of atomic weights 16, 17, and 18. Uranium, a very important element today, has two isotopes of atomic weights 238 and 235.

The startling fact which emerged from the discovery of isotopes is that when all isotopes are distinguished and the relative weights of the distinct elements determined, the weight of any one element is within 1% of a whole number. Such a fact can hardly be accidental. The explanation would seem to be that all these elements are really multiples of a single element, namely the lighter isotope of hydrogen, which has the least weight of all elements. In other words, the various elements which previously appeared to be entirely different substances, now were seen to be just smaller or larger collections of the same element, but arranged in special ways peculiar to the substance. (Strictly speaking, the fundamental building block is not the lighter isotope of hydrogen, but what is now called the proton. The lighter hydrogen isotope also has an electron whose weight is insignificant by comparison.)

If all of the different elements are really just aggregates of the lighter hydrogen atom, it should be possible to remove some atoms and convert one substance into another. Thus, we should be able to convert mercury, which is the next heavier element after gold, into gold. And we can. What the medieval alchemists hoped to do on mystical and superficial grounds, we can now do on the basis of far better scientific knowledge. Unfortunately, the cost of converting mercury into gold is too great to make it worth while. But we do have uses for the transmutation of elements which are, in our age, more valued, and which we shall describe in a moment.

A scientist who has a theory cannot afford to overlook even one detail, trivial as it may seem, which does not square with his theory. If all elements are merely combinations of the lighter hydrogen atom, then their weights should be exact multiples of the weight of this atom instead of having values within 1% of such weights. (The electrons in the atoms do not account for the difference.) This discrepancy must be explained. The lightest isotope of oxygen had, somewhat arbitrarily, been given weight 16. With this rather arbitrary standard, the lighter hydrogen atom has weight 1.008 rather than exactly 1. But helium, which consists of 4 hydrogen atoms, proves to have weight 4.0028. However, if it consists of 4 hydrogen atoms, its atomic weight should be 4 times 1.008, or 4.032. The difference, 4.032 – 4.0028, or about 0.03, is the discrepancy which must be accounted for. Now it so happens that Einstein, working in an entirely different field, the theory of relativity, had already shown that mass can be converted into energy. Energy can take different forms. It can be the heat created by burning coal or wood, or it can be radiation such as comes to us from the sun. At the moment the precise form of it does not matter. What does matter is the thought which occurred to scientists that perhaps, when 4 hydrogen atoms are fused to form helium, the missing 0.03 of matter is converted into energy in the process. Hence the fusion of elements should release energy. And experiments showed that this is indeed what happens. The energy which is released is called the binding energy, and it is this energy which is released when a hydrogen bomb is exploded.

In this brief account of the role of arithmetic in chemistry and atomic theory, we have said almost nothing about the great thinking and brilliant experiments which physicists and chemists contributed. Our interest has been to show how the use of simple numbers supplies scientists with a powerful tool. Of course, the mathematics of numbers remains to be developed, and we shall learn how much more can be accomplished with slightly more advanced tools. But we can already see something of what the Pythagoreans envisioned when they spoke of numbers as the essence of reality.

#### REVIEW EXERCISES

1. Calculate:

a) image

b) image

c) image

d) image

e) image

f) image

g) image

h) image

i) image

j) image

k) image

2. Calculate:

a) image

b) image

c) image

d) image

e) image

f) image

g) image

h) image

i) image

j) image

k) image

l) image

m) image

n) image

o) −8 ÷ −2

3. Calculate:

a) (2 · 5) (2 · 7)

b) 2a · 2b

c) 2a · 3b

d) 2x · 3y

e) 2x · 3y · 4z

4. Calculate:

a) image

b) image

c) image

d) image

e) image

5. Calculate:

a) image

b) image

c) image

d) image

e) image

f) image

g) image

h) image

6. Simplify:

a) image

b) image

c) image

d) image

e) image

f) image

g) image

h) image

7. Write as a fraction:

a) 0.294

b) 0.3742

c) 0.08

d) 0.003

8. Approximate by a number which is correct to 1 decimal place:

a) image

b) image

c) image

9. Are the following equations true for all values of a and b? [Suggestion: If you wish to disprove a general statement, it is sufficient to show one instance where it does not hold.]

a) 2(a + b) = 2a + 2b

b) 2ab = 2a · 2b

c) image

d) image

e) image

f) image

g) image

h) image

i) image

10. Write the following numbers in positional notation but in base 2. The only digits one can use in base 2 are 0 and 1.

a) 1

b) 3

c) 5

d) 7

e) 8

f) 16

g) 19

11. The following numbers are in base 2. Write the corresponding quantities in base 10.

a) 1

b) 101

c) 110

d) 1101

e) 1001

##### Topics for Further Investigation

1. The Egyptian method of writing whole numbers and fractions.

2. The Babylonian method of writing whole numbers and fractions.

3. The Roman method of writing whole numbers and fractions.

4. The fundamental arithmetical laws of atomic theory. (Use the references to Holton and Roller and to Bonner and Phillips).

5. Pythagorean number theory.

##### Recommended Reading

BALL, W. W. ROUSE: A Short Account of the History of Mathematics, Chaps. 1 and 2, Dover Publications, Inc., New York, 1960.

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[\*](#ipg82fn1) Starred sections and chapters can be omitted without disrupting the mathematical continuity.